Electromagnetic Waves in a cold, magnetized plasma

We have discussed the propagation of electromagnetic waves in an unmagnetized plasma using a kinetic approach. In the case of a magnetized plasma such a treatment quickly becomes very complicated due to the gyration of particles around the local magnetic field.

Instead we will carry out a cold fluid treatment. This approach is typically valid if

$$\frac{\partial}{\partial t} \gg V_t \nabla$$

$$\Rightarrow \quad \omega \gg k V_t$$

Assume that have a uniform magnetic field.

$$\underset{\sim}{B_0} = B_0 \hat{z}$$

and uniform ρ_0, P_0 etc. with $\underset{\sim}{u_0} = 0$

Assume linear perturbations of the form

$$\underset{\sim}{u_1} = \mathrm{Re}\left(\hat{\underset{\sim}{u}}\, e^{i \underset{\sim}{k} \cdot \underset{\sim}{x} - i\omega t} \right) \quad \Rightarrow \text{both species}$$

Momentum eqn :

nonlinear

$$\frac{\partial}{\partial t}\underline{u}_1 + \cancel{\underline{u}_1\cdot\nabla\underline{u}_1} = \frac{q}{m}\underline{E}_1 + \frac{q}{mc}\underline{u}_1\times\underline{B}_0$$

$$-i\omega\hat{\underline{u}} = \frac{q}{m}\hat{\underline{E}} + \Omega\,\hat{\underline{u}}\times\hat{z} \qquad \Omega = \frac{qB_0}{mc}$$

$$\hat{\underline{u}} = \hat{u}_\parallel\hat{z} + \underline{u}_\perp$$

$$\boxed{-i\omega\hat{u}_\parallel = \frac{q}{m}\hat{E}_z} \quad \Rightarrow \text{not affected by } B_0$$

$$-i\omega\hat{\underline{u}}_\perp = \frac{q}{m}\hat{\underline{E}}_\perp + \Omega\,\hat{\underline{u}}_\perp\times\hat{z}$$

$$\hat{\underline{u}}_\perp = \frac{1}{-i\omega}\left(\frac{q}{m}\hat{\underline{E}}_\perp + \Omega\,\hat{\underline{u}}_\perp\times\hat{z}\right)$$

$$-i\omega\hat{\underline{u}}_\perp = \frac{q}{m}\hat{\underline{E}}_\perp + \frac{\Omega}{-i\omega}\left(\frac{q}{m}\underline{E}_\perp\times\hat{z} + \Omega\underbrace{(\hat{\underline{u}}_\perp\times\hat{z})\times\hat{z}}_{-\hat{u}_\perp}\right)$$

$$(\Omega^2-\omega^2)\hat{\underline{u}}_\perp = -i\omega\frac{q}{m}\hat{\underline{E}}_\perp + \Omega\frac{q}{m}\underline{E}_\perp\times\hat{z}$$

$$\hat{\underline{u}}_\perp = \frac{1}{\omega^2-\Omega^2}\left[i\omega\frac{q}{m}\hat{\underline{E}}_\perp - \frac{q}{m}\Omega\,\hat{\underline{E}}_\perp\times\hat{z}\right]$$

<u>limits</u>

$$\Omega\to 0 \qquad \hat{\underline{u}}_\perp \simeq \frac{q}{m}\frac{\hat{\underline{E}}_\perp}{-i\omega}$$

$$\Omega \gg \omega \qquad \hat{\underline{u}}_\perp \simeq \frac{c}{B_0}\hat{\underline{E}}_\perp\times\hat{z} - \frac{i\omega c\hat{\underline{E}}_\perp}{\Omega B_0} \Rightarrow \underline{E}\times\underline{B} \text{ drift} + \text{pol. drift}$$

Ampere's Law

$$i\underset{\sim}{k} \times \underset{\sim}{\hat{B}} = \frac{4\pi}{c}\underset{\sim}{\hat{J}} - \frac{i\omega}{c}\underset{\sim}{\hat{E}}$$

$$\underset{\sim}{\hat{J}} = \sum_\alpha n_0 q_\alpha \underset{\sim}{\hat{u}}_\alpha$$

$$4\pi\underset{\sim}{\hat{J}} = \sum_\alpha \frac{4\pi n_0 q_\alpha^2}{m_\alpha}\left[\frac{1}{\omega^2-\Omega_\alpha^2}\left(i\omega\underset{\sim}{\hat{E}}_\perp - \Omega_\alpha \underset{\sim}{\hat{E}}_\perp \times \hat{z}\right) - \frac{1}{i\omega}\hat{z}\,\hat{z}\cdot\underset{\sim}{\hat{E}}\right]$$

$$= \sum_\alpha \omega_{p\alpha}^2\left[\frac{i\omega}{\omega^2-\Omega_\alpha^2}\underset{\sim}{\hat{E}}_\perp - \frac{\Omega_\alpha}{\omega^2-\Omega_\alpha^2}\underset{\sim}{\hat{E}}_\perp\times\hat{z} - \frac{1}{i\omega}\hat{z}\,\hat{z}\cdot\underset{\sim}{\hat{E}}\right]$$

$$i\underset{\sim}{k}\times\underset{\sim}{\hat{B}} = -\frac{i\omega}{c}\left[\hat{E} - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2-\Omega_\alpha^2}\underset{\sim}{\hat{E}}_\perp - i\sum_\alpha \frac{\frac{\omega_{p\alpha}^2\Omega_\alpha}{\omega}}{\omega^2-\Omega_\alpha^2}\underset{\sim}{\hat{E}}_\perp\times\hat{z} - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2}\hat{z}\,\hat{z}\cdot\underset{\sim}{\hat{E}}\right]$$

$$\equiv -i\frac{\omega}{c}\underset{\approx}{\epsilon}\cdot\underset{\sim}{\hat{E}} \qquad \Rightarrow \underset{\approx}{\epsilon} = \text{dielectric tensor}$$

$$\underset{\approx}{\epsilon} = \begin{bmatrix} \epsilon_\perp & -i\epsilon_x & 0 \\ i\epsilon_x & \epsilon_\perp & 0 \\ 0 & 0 & \epsilon_\parallel \end{bmatrix}$$

$$\epsilon_\perp = 1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2 - \Omega_\alpha^2}$$

$$\epsilon_\parallel = 1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2}$$

$$\epsilon_x = \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2 - \Omega_\alpha^2} \frac{\Omega_\alpha}{\omega}$$

note that $(\underset{\approx}{\epsilon}^t)^* = \underset{\approx}{\epsilon}$

$\Rightarrow$ hermitian

Faraday's Law

$$-i\frac{\omega}{c}\hat{\underset{\sim}{B}} + i\,\underset{\sim}{k} \times \hat{\underset{\sim}{E}} = 0 \qquad \Rightarrow \quad \hat{\underset{\sim}{B}} = \frac{c}{\omega}\underset{\sim}{k} \times \hat{\underset{\sim}{E}}$$

from Ampere's law

$$\underset{\sim}{k} \times \frac{c}{\omega}(\underset{\sim}{k} \times \hat{\underset{\sim}{E}}) = -\frac{\omega}{c}\underset{\approx}{\epsilon} \cdot \underset{\sim}{E}$$

$$\left(\underset{\approx}{I}\frac{c^2}{\omega^2}k^2 - \frac{c^2}{\omega^2}\underset{\sim}{k}\underset{\sim}{k} - \underset{\approx}{\epsilon}\right) \cdot \hat{\underset{\sim}{E}} = 0$$

$$\det\left[\frac{c^2 k^2}{\omega^2}\underset{\approx}{I} - \frac{c^2}{\omega^2}\underset{\sim}{k}\underset{\sim}{k} - \underset{\approx}{\epsilon}\right] = 0$$

$\Rightarrow$ determines $\omega(\underset{\sim}{k})$

Easiest to consider the two special cases of the dispersion relation for $\underset{\sim}{k}$ parallel and perpendicular to the magnetic field.

Case I : $\underset{\sim}{k}$ parallel to $\underset{\sim}{B}$

$$\begin{vmatrix} \frac{k_\parallel^2 c^2}{\omega^2} - \epsilon_\perp & i\epsilon_X & 0 \\ -i\epsilon_X & \frac{k_\parallel^2 c^2}{\omega^2} - \epsilon_\perp & 0 \\ 0 & 0 & -\epsilon_\parallel \end{vmatrix} = 0$$

~~three solutions~~

$$\epsilon_\parallel \left[\left(\frac{k_\parallel^2 c^2}{\omega^2} - \epsilon_\perp \right)^2 - \epsilon_X^2 \right] = 0$$

$\Rightarrow$ three solutions

$\epsilon_\parallel = 0$ e.s. wave

$\frac{k_\parallel^2 c^2}{\omega^2} - \epsilon_\perp = \pm \epsilon_X$ em waves polarization $\Rightarrow$ what is the direction of $\hat{\underset{\sim}{E}}$?

$$\left.\begin{aligned}&\left(\frac{k_{\parallel}^2 c^2}{\omega^2} - \epsilon_{\perp}\right)\hat{E}_x + i\epsilon_x \hat{E}_y = 0\\ &\epsilon_{\parallel}\hat{E}_z = 0\end{aligned}\right\}$$

(1) $\epsilon_{\parallel} = 0 \Rightarrow \hat{E}_z \neq 0$, $\hat{E}_x, \hat{E}_y = 0$

$\Rightarrow \quad 1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2} = 0$

$\Rightarrow$ electrostatic wave $\Rightarrow$ $\hat{\underset{\sim}{E}}$ parallel to $\underset{\sim}{k}$

$\Rightarrow$ no dependence on $\underset{\sim}{B}$ since motion is along $\underset{\sim}{B}$. $\Rightarrow$ same as $\underset{\sim}{B} = 0$ case.

(2) $\frac{k_{\parallel}^2 c^2}{\omega^2} - \epsilon_{\perp} = \pm\epsilon_x$, $\hat{E}_z = 0$

$$\pm\cancel{\epsilon_x}\hat{E}_x + i\cancel{\epsilon_x}\hat{E}_y = 0$$

$$\frac{\hat{E}_y}{\hat{E}_x} = \pm i$$

$\Rightarrow$ circularly polarized wave

$$E_x = \mathrm{Re}\left(\hat{E}_x e^{ikz - i\omega t}\right)$$

$$= \hat{E}_x \cos(kz - \omega t) \qquad \text{take } \hat{E}_x \text{ real}$$

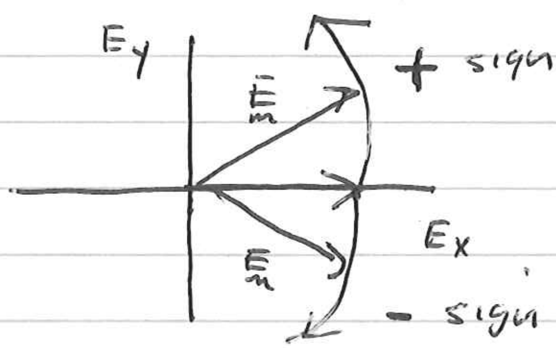
$$E_y = \mathrm{Re}\left(\pm i \hat{E}_x e^{ikz - i\omega t}\right)$$

$$= \mp \hat{E}_x \sin(kz - \omega t)$$

look at $z = 0$

$E_x \sim \cos\omega t$

$E_y \sim \pm \sin\omega t$



Compare with electron/ion rotation in $\underset{\sim}{B}$

$$m \frac{d}{dt} v_x = q \frac{1}{c} v_y B_0 \Rightarrow \dot{v}_x = \Omega v_y$$

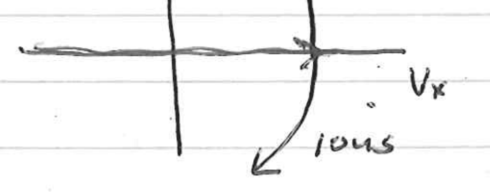
$$\cancel{m \frac{d}{dt} v_y = -\frac{q}{c} v_x B_0 \Rightarrow \dot{v}_y = -\Omega v_x}$$

$$v_x = v_0 \cos\Omega t$$

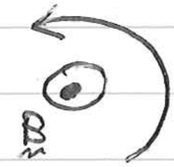
$$v_y = -v_0 \sin\Omega t$$

$\Omega > 0$ ions

$\Omega < 0$ electrons



Electrons rotate in the "right hand" sense with respect to $\underset{\sim}{B}$.



Ions rotate in the "left hand" sense.

$\Rightarrow$ wave with $+$ rotates in electron direction

$\Rightarrow$ wave with $-$ rotates in ion direction

~~dispersion~~
~~Electromag~~

$$\frac{k_\parallel^2 c^2}{\omega^2} = 1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2 - \Omega_\alpha^2} \pm \sum_\alpha \frac{\Omega_\alpha}{\omega} \frac{\omega_{p\alpha}^2}{\omega^2 - \Omega_\alpha^2}$$

$$= 1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega(\omega \pm \Omega_\alpha)}$$

$$= 1 - \frac{\omega_{pi}^2}{\omega(\omega \pm \Omega_i)} - \frac{\omega_{pe}^2}{\omega(\omega \mp \Omega_e)}$$

Ω_i, Ω_e now positive

$+$ resonates with electrons

$-$ resonates with ions

Consider + root $\longrightarrow$ right hand wave

$$\frac{k_\parallel^2 c^2}{\omega^2} = 1 - \frac{\omega_{pi}^2}{\omega(\omega+\Omega_i)} - \frac{\omega_{pe}^2}{\omega(\omega-\Omega_e)}$$

$\boxed{\omega \ll \Omega_i, \Omega_e}$

$$\frac{k_\parallel^2 c^2}{\omega^2} = 1 - \frac{\omega_{pi}^2}{\omega\Omega_i}\left(\cancel{1} - \frac{\omega}{\Omega_i}\right) + \frac{\omega_{pe}^2}{\omega\Omega_e}\left(\cancel{1} + \frac{\omega}{\Omega_e}\right)$$

$$= 1 + \frac{\omega_{pi}^2}{\Omega_i^2} + \frac{\omega_{pe}^2}{\Omega_e^2} \qquad \omega_{pe} \sim \Omega_e$$

$$\frac{k_\parallel^2 c^2}{\omega^2} \simeq \frac{\omega_{pi}^2}{\Omega_i^2} \sim \frac{m_i}{m_e} \quad \omega^2 = k_\parallel^2 \cancel{c^2} \frac{\cancel{e^2} B^2 \, \cancel{m_i}}{m_i^{\cancel{2}} \cancel{c^2} \, 4\pi n \cancel{e^2}} = k_\parallel^2 C_A^2$$

$\Longrightarrow$ Alfven waves

$\boxed{\Omega_i \ll \omega \ll \Omega_e, \omega_{pe}}$ $\qquad \omega_{pe} \sim \Omega_e$

$$\frac{k_\parallel^2 c^2}{\omega^2} = 1 - \frac{\omega_{pi}^2}{\omega^2} + \frac{\omega_{pe}^2}{\omega\Omega_e}$$

$$\frac{\omega_{pe}^2}{\omega\Omega_e} \sim \frac{\Omega_e}{\omega} \gg 1$$

$$\frac{\omega_{pi}^2}{\omega^2} \sim \frac{m_e}{m_i}\frac{\Omega_e^2}{\omega^2} \sim \frac{\Omega_e\Omega_i}{\omega^2} \ll \frac{\Omega_e}{\omega}$$

$$\frac{k_\parallel^2 c^2}{\omega^{\cancel{2}}} = \frac{\omega_{pe}^2}{\cancel{\omega}\,\Omega_e} \qquad \frac{\omega}{\Omega_e} \sim k_\parallel^2 \frac{c^2}{\omega_{pe}^2}$$

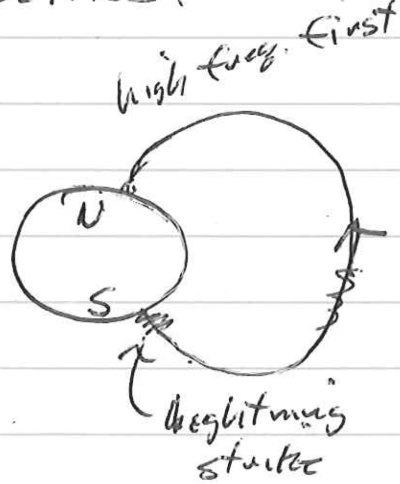
whistler wave

⇒ this is a dispersive wave in which only electrons contribute

⇒ rotates in electron sense.

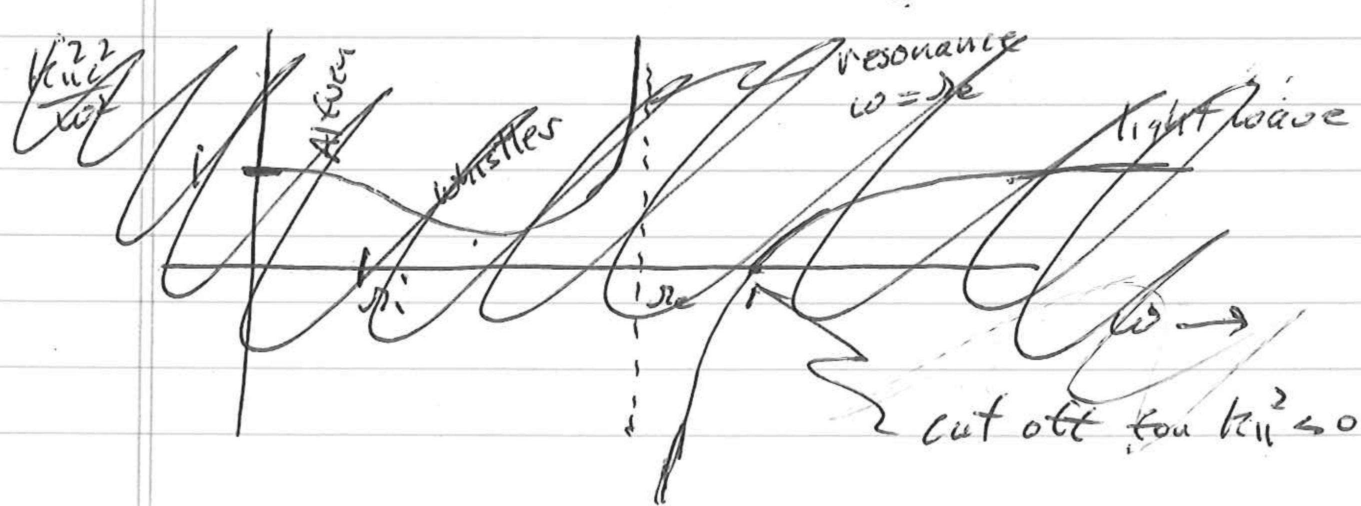
⇒ short wavelengths have higher group and phase velocities.

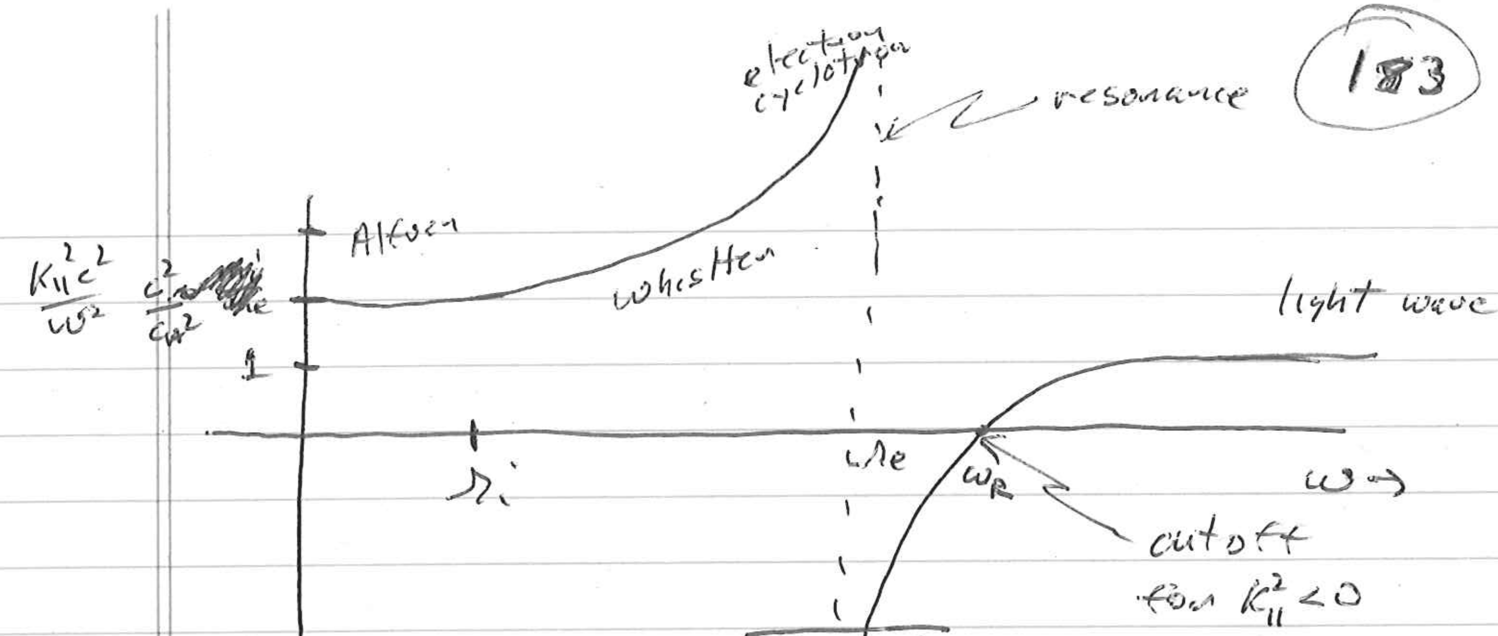
~~⇒ sounds~~



⇒ high freq then low freq. sounds like whistle

$$\frac{k_\parallel^2 c^2}{\omega^2} = 1 - \frac{\omega_{pi}^2}{\omega(\omega+\Omega_i)} - \frac{\omega_{pe}^2}{\omega(\omega-\Omega_e)}$$



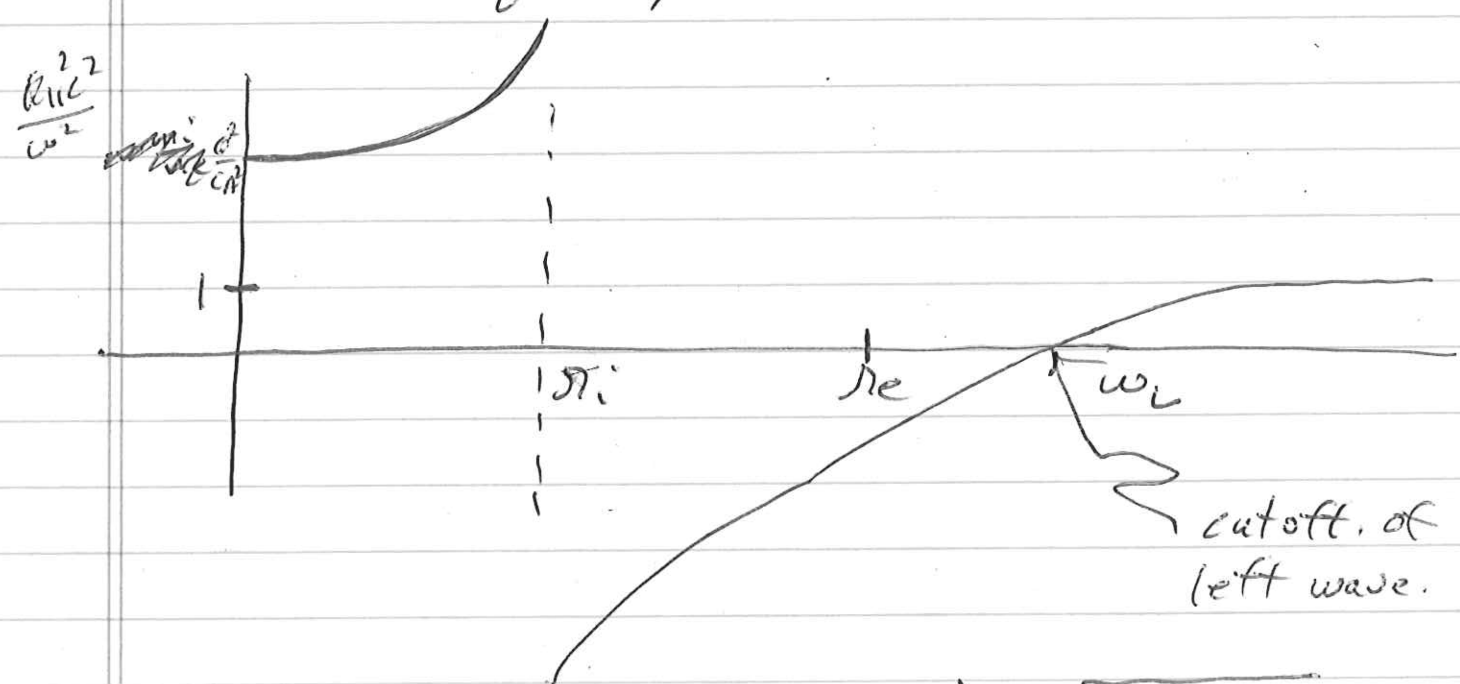


$$1 - \frac{\omega_{pe}^2}{\omega_R(\omega_R - \Omega_e)} = 0 \qquad \omega_R = \frac{\Omega_e + \sqrt{\Omega_e^2 + 4\omega_{pe}^2}}{2}$$

Consider − root ⇒ left hand wave

$$\frac{k_\parallel^2 c^2}{\omega^2} = 1 - \frac{\omega_{pi}^2}{\omega(\omega - \Omega_i)} - \frac{\omega_{pe}^2}{\omega(\omega + \Omega_e)}$$

⇒ again have Alfven wave at low frequency



$$\omega_L = \frac{-\Omega_e + \sqrt{\Omega_e^2 + 4\omega_{pe}^2}}{2}$$

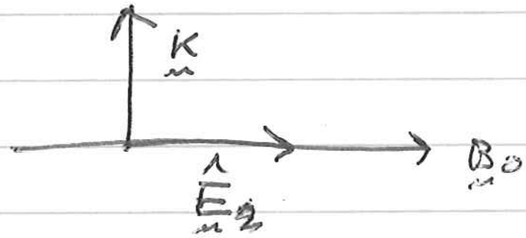
<u>Case II</u> : Propagation $\perp$ to $\underset{\sim}{B}$ $\Rightarrow$ Let $k = k_x$

$$\begin{vmatrix} -\epsilon_\perp & i\epsilon_X & 0 \\ -i\epsilon_X & \frac{k_\perp^2 c^2}{\omega^2} - \epsilon_\perp & \\ 0 & & \frac{k_\perp^2 c^2}{\omega^2} - \epsilon_\parallel \end{vmatrix} = 0$$

$$\left(\frac{k_\perp^2 c^2}{\omega^2} - \epsilon_\parallel\right)\left(-\epsilon_\perp\left(\frac{k_\perp^2 c^2}{\omega^2} - \epsilon_\perp\right) - \epsilon_X^2\right) = 0$$

(1) $\frac{k_\perp^2 c^2}{\omega^2} = 1 - \frac{\omega_{pe}^2}{\omega^2}$, $E_z \neq 0$, $E_x, E_y = 0$

$$\omega^2 = k_\perp^2 c^2 + \omega_{pe}^2$$



$\hat{\underset{\sim}{E}}$ points along $\underset{\sim}{B}$ so
motion unaffected by $\underset{\sim}{B}$.

$\Longrightarrow$ "ordinary mode"

(2) $\epsilon_\perp\left(\frac{k_\perp^2 c^2}{\omega^2} - \epsilon_\perp\right) = -\epsilon_X^2$

$$\boxed{\frac{k_\perp^2 c^2}{\omega^2} = \frac{(\epsilon_\perp - \epsilon_X)(\epsilon_\perp + \epsilon_X)}{\epsilon_\perp}}$$

~~"ex" for ordinary mode" X-mode~~

low frequency limit $\omega \ll \Omega$

$$\epsilon_\perp \simeq 1 + \frac{\omega_{pi}^2}{\Omega_i^2} + \frac{\omega_{pe}^2}{\Omega_e^2}$$

$$\simeq \frac{\omega_{pi}^2}{\Omega_i^2}$$

$$\epsilon_x = -\sum_\alpha \frac{\omega_{p\alpha}^2}{\Omega_\alpha} \frac{1}{\omega} = \left(-\frac{\omega_{pi}^2}{\Omega_i} + \frac{\omega_{pe}^2}{\Omega_e} \right) \frac{1}{\omega}$$

$$= 0$$

$$\frac{k_\perp^2 c^2}{\omega^2} \simeq \epsilon_\perp = \frac{\omega_{pi}^2}{\Omega_i^2} \Rightarrow \omega^2 = k_\perp^2 c_A^2$$

cold plasma limit of magnetosonic wave (fast mode).

Resonances $\quad \epsilon_\perp = 0$

Cutoffs $\quad \epsilon_\perp = \pm \epsilon_x$

Resonances $\epsilon_\perp = 0$

$$1 - \frac{\omega_{pi}^2}{\omega^2 - \Omega_i^2} - \frac{\omega_{pe}^2}{\omega^2 - \Omega_e^2} = 0$$

high freq. root $\cancel{\omega \gg \Omega_i}$ $\omega \sim \Omega_e, \omega_{pe}$

$$1 - \frac{\omega_{pe}^2}{\omega^2 - \Omega_e^2} = 0 \Rightarrow \boxed{\omega^2 = \Omega_e^2 + \omega_{pe}^2}$$

upper hybrid resonance

low freq. root $\omega \ll \Omega_e, \omega_{pe}$

$$1 - \frac{\omega_{pi}^2}{\omega^2 - \Omega_i^2} + \frac{\omega_{pe}^2}{\Omega_e^2} = 0$$

for ~~ω_{pi}~~ $\omega_{pe} \sim \Omega_e$, have $\omega_{pi} \gg \Omega_i$

$$\omega \sim \omega_{pi} \gg \Omega_i$$

$$1 - \frac{\omega_{pi}^2}{\omega^2} + \frac{\omega_{pe}^2}{\Omega_e^2} = 0$$

$$\boxed{\omega^2 = \frac{\omega_{pi}^2}{1 + \frac{\omega_{pe}^2}{\Omega_e^2}}}$$

lower hybrid resonance.
for $\omega_{pe} \gg \Omega_e \Rightarrow \sqrt{\Omega_e \Omega_i} \sim \omega$

ions behave like unmagnetized ($\omega \gg \Omega_i$)
electrons are strongly magnetized.
(ExB and pol. drift)

~~Rec~~ Wave polarization near a resonance.
In general have

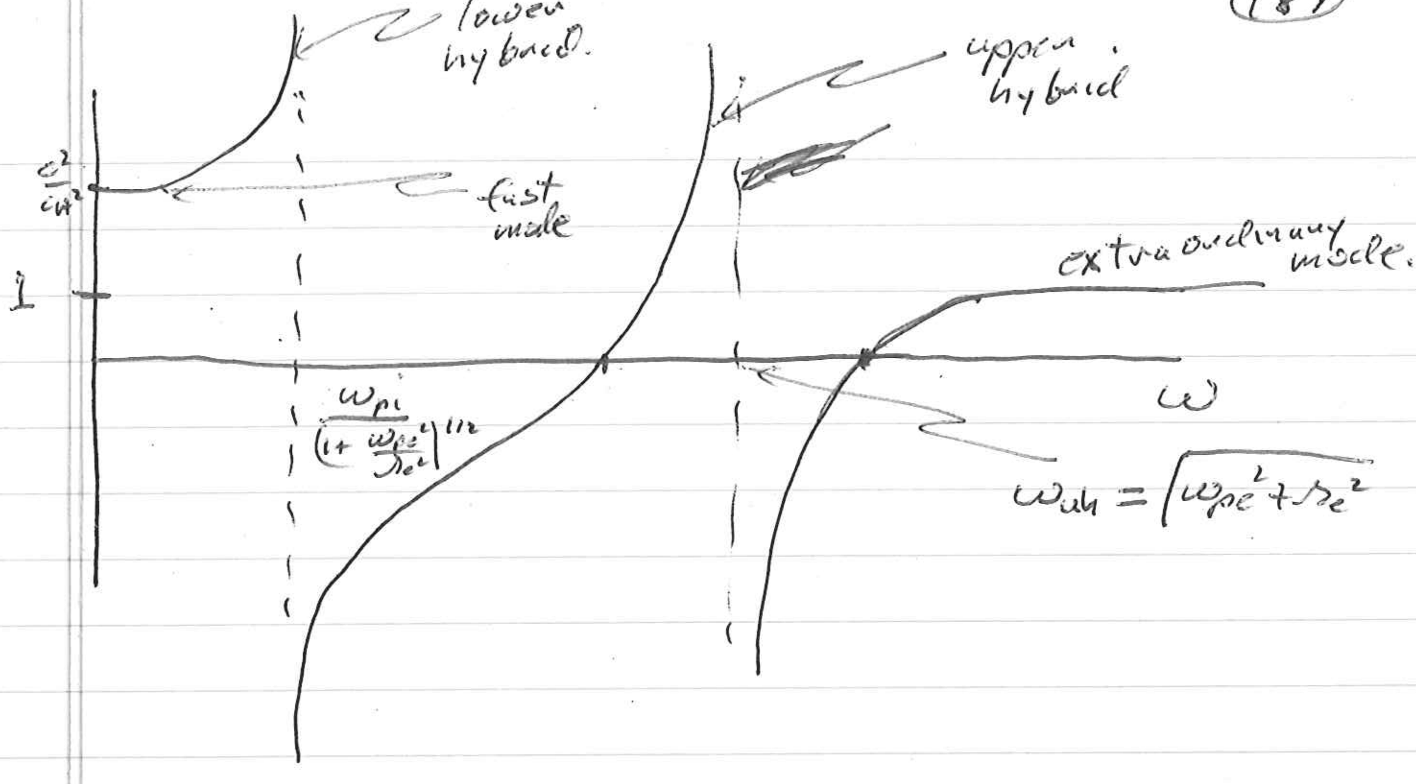
$$-\epsilon_\perp \hat{E}_x + i \epsilon_x \hat{E}_y = 0$$

$$\hat{E}_y = + \frac{\epsilon_\perp \hat{E}_x}{i \epsilon_x} \rightarrow 0 \quad \text{at a resonance}$$

$\Rightarrow \underset{\sim}{k} \times \underset{\sim}{E} \simeq 0$ since $\underset{\sim}{k} = k_\perp \hat{x}$

$\Rightarrow \underset{\sim}{\hat{B}} \rightarrow 0$ at resonce

$\Rightarrow$ waves become electrostatic



At a cutoff. $\epsilon_\perp = \pm \epsilon_x$

$$\hat{E}_y = \frac{\epsilon_\perp}{i\epsilon_x}\hat{E}_x = \mp i \hat{E}_x$$

$\Longrightarrow$ circularly polarized.