Double Adiabatic Equation
(CGL - Chew, Goldberger, Low)

What model do we use if the collisions are not strong enough to make the pressure tensor isotropic? The CGL model is sometimes useful if a full kinetic treatment is too complex. Start with the collisionless Boltzmann equation

$$\frac{\partial f}{\partial t} + v\cdot\nabla f + \frac{q}{m}\left(\underset{\sim}{E} + \frac{1}{c}\underset{\sim}{v}\times\underset{\sim}{B}\right)\cdot\frac{\partial}{\partial \underset{\sim}{v}} f = 0$$

As in MHD equations, want to order the electric and magnetic forces to be large. Thus, to lowest order we have

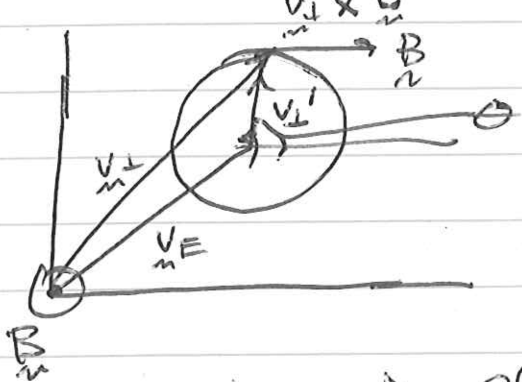
$$\frac{q}{m}\left(\underset{\sim}{E} + \frac{1}{c}\underset{\sim}{v}\times\underset{\sim}{B}\right)\cdot\frac{\partial f^{0}}{\partial \underset{\sim}{v}} = 0$$

Can get rid of $\underset{\sim}{E}$ by changing frames to remove the $\underset{\sim}{E}\times\underset{\sim}{B}$ drift.

$$\underset{\sim}{v}' = v - \frac{c}{B^2}\underset{\sim}{E}\times\underset{\sim}{B}$$

$$\Rightarrow \quad \frac{q}{m}\frac{1}{c}\underset{\sim}{v}'\times\underset{\sim}{B}\cdot\frac{\partial}{\partial \underset{\sim}{v}} f^{(0)} = 0$$

where we have taken $E_{\parallel} = 0$.



$\underset{\sim}{v} \cdot \underset{\sim}{B} = 0$ plane

$$\underset{\sim}{v}' \times \underset{\sim}{B} \cdot \frac{\partial}{\partial \underset{\sim}{v}} f^{(0)} = 0$$

implies that $f^{(0)}$ is const around a ~~circle~~ circle of constant $v_\perp'$

$$f^{(0)}(\underset{\sim}{v}, \underset{\sim}{x}, t) = f^{0}(v_\perp', v_\parallel, x, t)$$

In a local magnetic coordinate system, the parallel pressure decouples from the transverse pressure

$$P_\parallel = \int d\underset{\sim}{v}\, f^{(0)} (v_\parallel - \langle v_\parallel \rangle)^2\, m$$

$$P_{\perp 1} = m \int d\underset{\sim}{v}\, f^{(0)}\, v_\perp'^2 \underbrace{\cos^2\theta}_{\frac{1}{2}}$$

$$P_{\perp 2} = m \int d\underset{\sim}{v}\, f^{(0)}\, v_\perp'^2 \underbrace{\sin^2\theta}_{\frac{1}{2}}$$

$P_{\perp 1} = P_{\perp 2} = P_\perp$

Define the pressure tensor as
$$\begin{pmatrix} P_\perp & & 0 \\ & P_\perp & \\ 0 & & P_\parallel \end{pmatrix}$$

In vector form

$$\underset{\approx}{P} = P_{\parallel} \underset{\sim}{b}\underset{\sim}{b} + P_{\perp} \left(\underset{\approx}{I} - \underset{\sim}{b}\underset{\sim}{b} \right)$$

If we can determine $P_{\parallel}$ and $P_{\perp}$, we can construct a set of closed fluid equations. Can use the adiabatic variables to do this.
First have μ invariance.

$$\frac{V_{\perp}^2}{B} \sim \text{const}$$

Average over $V_{\perp}^2$

$$\frac{\langle V_{\perp}^2 \rangle}{B} = \text{const} \sim \frac{T_{\perp}}{B}$$

$$\sim \frac{P_{\perp}}{\rho B}$$

$$\boxed{\frac{d}{dt} \frac{P_{\perp}}{\rho B} = 0}$$ from μ invariance

From the first adiabatic invariant we can determine an equation for $P_{\parallel}$.

$$J_{\parallel} \sim \int dl \, V_{\parallel}$$

Consider a flux tube of length L, area A and local magnetic field B. We then have

① $\langle V_{\parallel}^2\rangle L^2 \sim \text{const}$ $J_{\parallel}$ invariant.

② $BA \sim \text{const}$ flux conservation

③ $\rho AL \sim \text{const}$ particle number conservation

Using ③ to get rid of L in ①, we have

$$\frac{\langle V_{\parallel}^2\rangle}{A^2\rho^2} \sim \text{const}$$

Using flux cons to get rid of A

$$\frac{\langle V_{\parallel}^2\rangle B^2}{\rho^2} \sim \text{const}$$

Get rid of B using μ invariance

$$\frac{\langle V_{\parallel}^2\rangle\langle V_{\perp}^4\rangle}{\rho^2} \sim \text{const}$$

$$\Longrightarrow \frac{T_{\parallel}T_{\perp}^2}{\rho^2} \sim \text{const}$$

$$\Longrightarrow \frac{P_{\parallel}P_{\perp}^2}{\rho^5} \sim \text{const}$$

$$\boxed{\frac{d}{dt}\left(\frac{P_{\parallel}P_{\perp}^2}{\rho^5}\right) = 0}$$

We now have equations of state for $P_{\parallel}, P_{\perp}$ and can write down the complete set of CGL equations.

$$\rho \frac{d}{dt} \underset{\sim}{u} = - \nabla \cdot \left(P_{\parallel} \underset{\sim}{b}\underset{\sim}{b} + P_{\perp} (\underset{\sim}{I} - \underset{\sim}{b}\underset{\sim}{b}) \right)$$

$$- \nabla \frac{B^2}{8\pi} + \frac{1}{4\pi} \underset{\sim}{B} \cdot \nabla \underset{\sim}{B}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \underset{\sim}{u} = 0$$

$$\frac{d}{dt} \left(\frac{P_{\perp}}{\rho B} \right) = 0$$

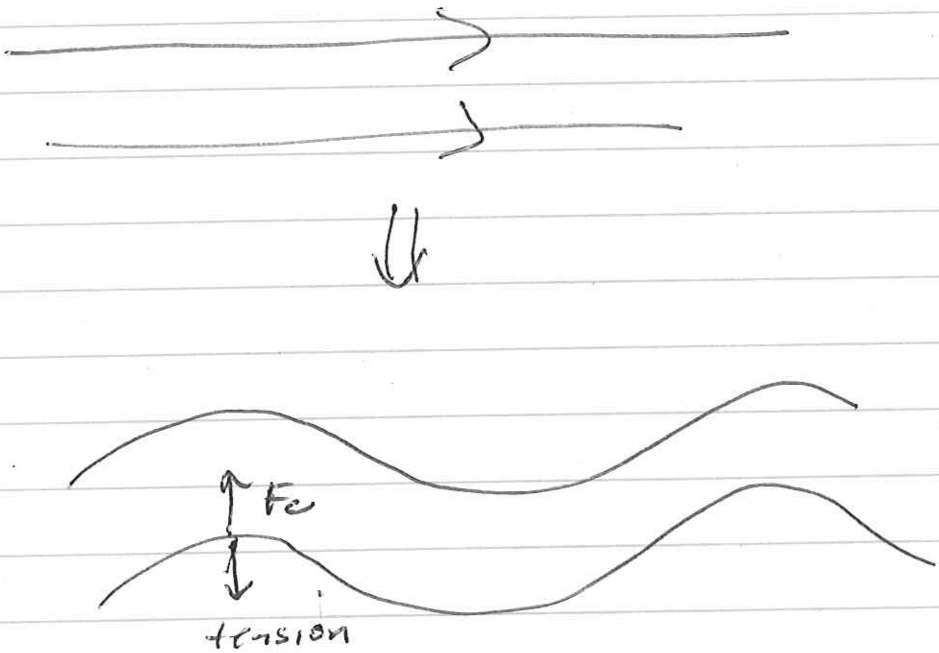
$$\frac{d}{dt} \left(\frac{P_{\parallel} P_{\perp}^2}{\rho^5} \right) = 0$$

$$\frac{\partial}{\partial t} \underset{\sim}{B} - \nabla \times (\underset{\sim}{u} \times \underset{\sim}{B}) = 0$$

A major limitation of these equations is that parallel transport is neglected

$\Rightarrow$ Calculate firehose instability using CGL model (Hwk)

Firehose basic physics



$$F_c \sim n_0 \frac{m\langle v_\parallel^2\rangle}{R} \sim m\langle v_\parallel^2\rangle K$$

$$F_t \sim \frac{B^2}{4\pi} K$$

Instability for $F_c > F_t$

$$n_0 m\langle v_\parallel^2\rangle > \frac{B^2}{4\pi}$$

$$\frac{n T_\parallel 4\pi}{B^2} > 1$$

$$\beta_\parallel > 1$$

Firehose instability using kinetic theory

Initial field $\underset{\sim}{B}_0 = B_0 \hat{z}$ with

$$f_0 = f_0(v_\perp, v_\parallel)$$

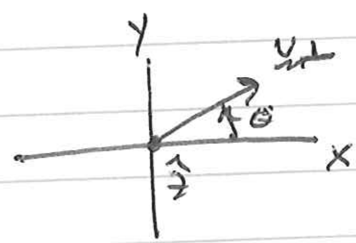
$$f_1 = \mathrm{Re}\left(\hat{f}\, e^{ikz - i\omega t}\right)$$

$$\underset{\sim}{B}_1 = (B_1, 0, 0)$$

$$\underset{\sim}{E}_1 = (0, E_1, 0)$$

$$\frac{1}{c}(-\cancel{i}\omega)\underset{\sim}{B}_1 + \cancel{i}k\, \hat{z} \times \underset{\sim}{E}_1 = 0$$

$$+\frac{\omega}{kc} B_1 + E_1 = 0$$



$$(-i\omega + ikv_\parallel)\hat{f} + \frac{q}{m}\left(\hat{E}\hat{y} + \frac{1}{c}\underset{\sim}{v} \times \hat{B}\hat{x}\right)\cdot\frac{\partial}{\partial \underset{\sim}{v}} f_0$$

$$+ \frac{q}{m}\frac{1}{c}\underbrace{\underset{\sim}{v} \times \hat{z} B_0 \cdot \frac{\partial}{\partial \underset{\sim}{v}}}_{-v_\perp \frac{1}{v_\perp}\frac{\partial}{\partial\theta}} \hat{f} = 0$$

$$\left(-i\omega + ikv_\parallel - \Omega_0\frac{\partial}{\partial\theta}\right)\hat{f} + \cancel{\frac{q}{m}\left(\hat{E}\hat{y}\cdot\frac{\partial}{\partial v} v_\perp^2\right.}$$

$$\cancel{+\frac{q}{m}\left(\hat{E}\hat{y}\cdot\frac{\partial}{\partial \underset{\sim}{v}} v_\perp^2 + \frac{1}{c}\hat{B}\,\underset{\sim}{v}\times\hat{x}\cdot\right.}$$

$$v_\perp^2 = v_x^2 + v_y^2$$

$$\underset{\sim}{v} \times \hat{x} \cdot \frac{\partial}{\partial \underset{\sim}{v}} f_0 = v_\parallel \, \hat{y} \cdot \frac{\partial}{\partial \underset{\sim}{v}} v_\perp^2 \frac{\partial}{\partial v_\perp^2} f_0$$

$$\cancel{+} \; v_y \, \hat{z} \cdot \frac{\partial}{\partial \underset{\sim}{v}} v_\parallel^2 \frac{\partial}{\partial v_\parallel^2} f_0$$

$$= 2 v_\parallel v_y \frac{\partial f_0}{\partial v_\perp^2} - 2 v_y v_\parallel \frac{\partial}{\partial v_\parallel^2} f_0$$

$$= 2 v_\parallel v_\perp \sin\theta \left(\frac{\partial f_0}{\partial v_\perp^2} - \frac{\partial f_0}{\partial v_\parallel^2} \right)$$

$$\left(-i\omega + i k v_\parallel - \Omega_0 \frac{\partial}{\partial \theta} \right) \hat{f}$$

$$+ \frac{q}{m} \left(\hat{E}\, 2 v_\perp \sin\theta \frac{\partial f_0}{\partial v_\perp^2} + \hat{B}\, 2 v_\parallel v_\perp \frac{\sin\theta}{c} \left(\frac{\partial f_0}{\partial v_\perp^2} - \frac{\partial f_0}{\partial v_\parallel^2} \right) \right)$$

$$\hat{B} = -\frac{kc}{\omega} \hat{E}$$

$$\left(-i\omega + i k v_\parallel - \Omega \frac{\partial}{\partial \theta} \right) \hat{f}$$

$$= -\frac{2q}{m} v_\perp \sin\theta \left[\left(\hat{E} + \frac{v_\parallel}{c} \hat{B} \right) \frac{\partial f_0}{\partial v_\perp^2} - \frac{v_\parallel}{c} \hat{B} \frac{\partial f_0}{\partial v_\parallel^2} \right]$$

$$= -\frac{2q}{m} v_\perp \sin\theta \, \hat{E} \left[\left(1 - \frac{k v_\parallel}{\omega} \right) \frac{\partial f_0}{\partial v_\perp^2} + \frac{k v_\parallel}{\omega} \frac{\partial f_0}{\partial v_\parallel^2} \right]$$

$$= S' \sin\theta$$

$$\hat{f} = \hat{f}_s \sin\theta + \hat{f}_c \cos\theta$$

$$\left(-i\omega + ikv_{\parallel}\right)\hat{f}_s + \Omega \hat{f}_c = S'$$

$$\left(-i\omega + ikv_{\parallel}\right)\hat{f}_c - \Omega \hat{f}_s = 0$$

$$i\underset{\sim}{k} \times \underset{\sim}{B}_1 = \frac{4\pi}{c}\underset{\sim}{J}_1$$

$$ikB_1 = \frac{4\pi}{c} J_{1y} = \frac{4\pi}{c} q \int d\underset{\sim}{v}\, f_1 v_\perp \sin\theta$$

$$= \frac{4\pi}{c} q \frac{1}{2} \int d\underset{\sim}{v}\, v_\perp f_{1s}$$

$$ik\hat{B} = \frac{4\pi}{c}\frac{q}{2}\int d\underset{\sim}{v}\, v_\perp \hat{f}_s$$

$\boxed{\text{Can show that } J_{1x} = 0}$ $\Rightarrow$ plane polarization OK

$$\left(-i\omega + ikv_{\parallel} + \frac{\Omega^2}{-i\omega + ikv_{\parallel}}\right)\hat{f}_s = S'$$

$\cancel{-i(\omega - kv_{\parallel})}$

$$\left[-(\omega - kv_{\parallel})^2 + \Omega^2\right]\hat{f}_s = S'\left(-i\omega + ikv_{\parallel}\right)$$

for $\omega, kv_{\parallel} \ll \Omega$

$$\hat{f}_s \simeq -i\frac{S'}{\Omega^2}(\omega - kv_{\parallel})$$

$$\hat{f}_s = -i\frac{\cancel{q}}{\Omega^2}\omega\left(1 - \frac{kv_{\parallel}}{\omega}\right)\left(-\frac{2q}{m}v_\perp \hat{E}\right)\left[\left(1 - \frac{k_{\parallel}v_{\parallel}}{\omega}\right)\frac{\partial f_0}{\partial v_\perp^2} + \frac{k_{\parallel}v_{\parallel}}{\omega}\frac{\partial f_0}{\partial v_{\parallel}^2}\right.$$

$$c_A^2 \,\cancel{i} k^{2} \cancel{\tilde{B}} = \cancel{\frac{4\pi}{c}} \cancel{\frac{n_0 q}{k}} \left(-\cancel{i}\frac{\omega}{\cancel{\Omega}}\right)\left(+\frac{\cancel{B_0}}{m}\right) \cancel{\hat{z}} \left(+\frac{\omega}{\cancel{\Omega}}\right) \cancel{\tilde{B}}\ \frac{m^2 c^2}{\cancel{4\pi} B_0^2}$$

$$\int d\underline{v}\ v_\perp^2 \left(1-\frac{k v_\parallel}{\omega}\right)\left[\left(1-\frac{k v_\parallel}{\omega}\right)\frac{\partial f_0}{\partial v_\perp^2} + \frac{k v_\parallel}{\omega}\frac{\partial f_0}{\partial v_\parallel^2}\right]\frac{1}{n_0}$$

$$-k^2 c_A^2 = \omega^2 \frac{1}{n_0}\int d\underline{v}\ v_\perp^2\left(1-\frac{k v_\parallel}{\omega}\right)\left[\left(1-\frac{k v_\parallel}{\omega}\right)\frac{\partial f_0}{\partial v_\perp^2} + \frac{k v_\parallel}{\omega}\frac{\partial f_0}{\partial v_\parallel^2}\right]$$

$$\langle v_\parallel \rangle = 0$$

$$-k^2 c_A^2 = \frac{\omega^2}{n_0}\int d\underline{v}\ v_\perp^2\left[\left(1+\frac{k^2 v_\parallel^2}{\omega^2}\right)\frac{\partial f_0}{\partial v_\perp^2} - \frac{k^2 v_\parallel^2}{\omega^2}\frac{\partial f_0}{\partial v_\parallel^2}\right]$$

$$\int d\underline{v}\ v_\perp^2 \frac{\partial f_0}{\partial v_\perp^2} = 2\pi \int_{-\infty}^{\infty} dv_\parallel \int_0^{\infty} \frac{dv_\perp^2}{2}\ v_\perp^2 \frac{\partial f_0}{\partial v_\perp^2}$$

$$= -\int d\underline{v}\ f_0$$

$$\int d\underline{v}\ v_\parallel^2 \frac{\partial f_0}{\partial v_\parallel^2} = 2\pi \int dv_\perp\, v_\perp\ dv_\parallel \frac{v_\parallel^{\cancel{2}}}{2 \cancel{v_\parallel}} \frac{\partial f_0}{\partial v_\parallel}$$

$$= -\frac{1}{2}\int d\underline{v}\ f_0$$

$$+k^2 c_A^2 = \frac{\omega^2}{n_0}\int d\underline{v}\ f_0\left[1 + \frac{k^2 v_\parallel^2}{\omega^2} - \frac{1}{2} v_\perp^2 \frac{k^2}{\omega^2}\right]$$

$$= \omega^2 + k^2 \langle v_\parallel^2 \rangle - \frac{1}{2} k^2 \langle v_\perp^2 \rangle$$

$$\langle v_\parallel^2 \rangle = \frac{1}{n_0}\int d\underline{v}\, f_0\, v_\parallel^2$$

$$= \int \frac{dv_\parallel}{\sqrt{\pi}\, v_{t\parallel}}\, v_\parallel^2\, e^{-v_\parallel^2/v_{\parallel t}^2}$$

$$= v_{\parallel t}^2 \int \frac{ds}{\sqrt{\pi}}\, s^2 e^{-s^2}$$

$$= -v_{\parallel t}^2 \frac{d}{d\epsilon}\frac{1}{\sqrt{\epsilon}} = \frac{1}{2} v_{\parallel t}^2 = \frac{1}{m} T_\parallel$$

$$\langle v_\parallel^2 \rangle = \frac{T_\parallel}{m}$$

$$\langle v_\perp^2 \rangle = \int d\underline{v}\, f_0\, v_\perp^2$$

$$= \int d\underline{v}\, f_0 \left(v_x^2 + v_y^2\right) = \frac{2}{m} T_\perp$$

$$k^2 c_A^2 = \omega^2 + k^2 \left(T_\parallel - T_\perp\right)\frac{1}{m}$$

$$\omega^2 = \cancel{k^2} k^2 c_A^2 \Bigg(1 - \underbrace{\frac{T_\parallel}{m c_A^2}}_{\underbrace{\frac{T_\parallel 4\pi m n_0}{m B^2}}_{\frac{\beta_\parallel}{2}}} + \frac{T_\perp}{m c_A^2}\Bigg)$$

$$\boxed{\omega^2 = k^2 c_A^2 \left(1 - \frac{\beta_\parallel}{2} + \frac{\beta_\perp}{2}\right)}$$

firehose instability