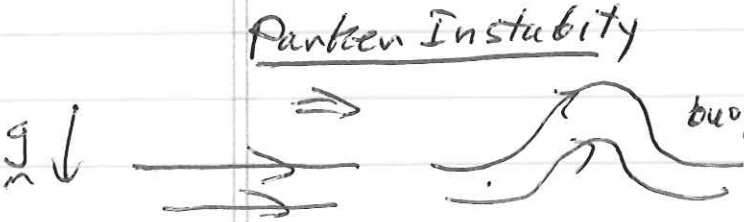
Ideal MHD stability

Equilibrium plasmas are subject to a wide variety of instabilities whose dynamics have important consequences for plasma **evolution**. An instability must have a source of free energy. Examples are

gravitational — Rayleigh-Taylor

⇒ heavy material supported by light material (Sun, shocks)

Parker Instability



Magneto-rotational inst.
⇒ angular momentum transport in accretion discs.

Flow — Kelvin Helmholtz

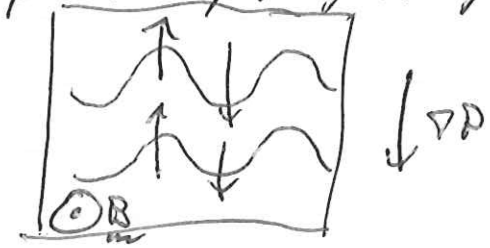
~~sto~~ solar wind - planetary magnetosphere

Magnetic Field — sausage

kink

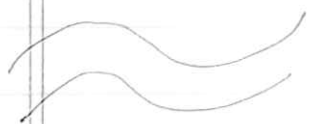
magnetic reconnection

Pressure — Interchange — fusion expts
planetary magnetospheres.



unfavorable curvature $\Rightarrow$ like RT

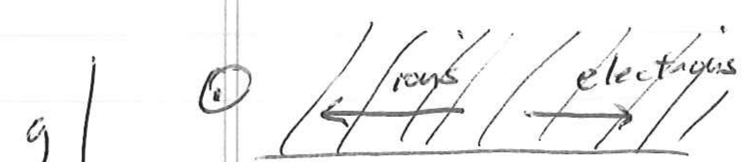
drift-waves — fusion, magnetospheres



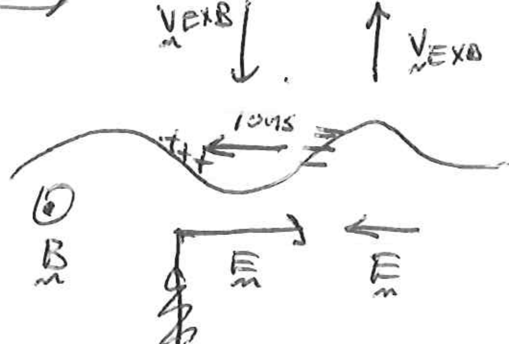
Firehose — $P_{\parallel} > P_{\perp}$ } $\underset{\sim}{B} \neq 0$
Mirror — $P_{\perp} > P_{\parallel}$
Weibel — $P_{xx} \neq P_{yy}$ } $\underset{\sim}{B} = 0$
shocks

To analyze stability must start with an ~~eqt~~ equilibrium and then linearize equations around that equilibrium. Will give several examples.

Rayleigh-Taylor instability



$\underset{\sim}{v} = \frac{c}{eB^2} \underset{\sim}{F} \times \underset{\sim}{B}$ light material e.g. $\underset{\sim}{B}$



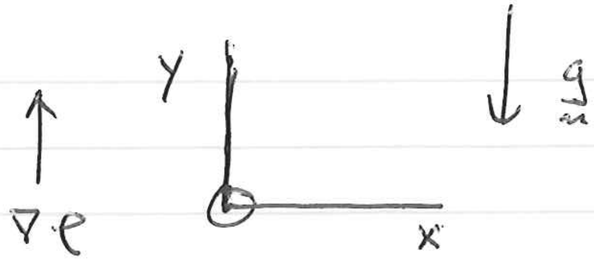
Electrons and ions drift

Rosenbluth/Longmire

$$0 = \rho \underset{\sim}{g} + \frac{\underset{\sim}{J} \times \underset{\sim}{B}}{c}$$

$$\underset{\sim}{J} = c\rho\, \underset{\sim}{g} \times \underset{\sim}{B} \frac{1}{B^2}$$

<u>equilibrium</u>



$\mathbf{B} = B_0(y)\hat{z}$

$P_0(y)$

$\rho_0(y)$

$$0 = -\frac{\partial}{\partial y}\left(P_0 + \frac{B_0^2}{8\pi}\right) - \rho_0 g$$

$$\frac{1}{\rho_0}\frac{d\rho_0}{dy} \sim \frac{1}{L} \quad \Rightarrow \text{ gradient scale length.}$$

scaling for growth rate

$$\gamma_g \sim \sqrt{\frac{g}{L}} \quad \Rightarrow \text{ dimensional argument.}$$

Another time scale of the system is the fast mode propagation time

$$\gamma_f = \frac{c_f}{L}$$

Assume weak gravitation

$$\gamma_g << \gamma_f$$

$\Rightarrow$ simplifies equations

$\Rightarrow$ nearly incompressible motion

Linearized equations

$$\rho_1(x,y,t) = \mathrm{Re}\left(\hat{\rho}(y)\, e^{ikx - i\omega t}\right)$$

$$(1) \quad \rho_0 \frac{\partial}{\partial t} \mathbf{u}_1 = -\nabla\left(P_1 + \frac{B_0 B_1}{4\pi}\right) - \rho_1 g \hat{y}$$

$$\frac{\partial}{\partial t} \mathbf{B}_1 \cancel{+} \nabla \times (\mathbf{u}_1 \times \mathbf{B}_0) = 0$$

$$\frac{\partial}{\partial t} \mathbf{B}_1 - \cancelto{0}{\mathbf{B}_0 \cdot \nabla \mathbf{u}_1} + \mathbf{B}_0 \nabla \cdot \mathbf{u}_1 + \mathbf{u}_1 \cdot \nabla \mathbf{B}_0 = 0$$

$$\mathbf{B}_1 = B_1 \hat{z}$$

$$(2) \quad \frac{\partial}{\partial t} B_1 + B_0 \nabla \cdot \mathbf{u}_1 + u_{1y} \frac{\partial}{\partial y} B_0 = 0$$

$$(3) \quad \frac{\partial}{\partial t} P_1 + u_{y1} \frac{\partial}{\partial y} P_0 + \Gamma P_0 \nabla \cdot \mathbf{u}_1 = 0$$

$$(4) \quad \frac{\partial}{\partial t} \rho_1 + u_{y1} \frac{\partial}{\partial y} \rho_0 + \rho_0 \nabla \cdot \mathbf{u}_1 = 0$$

$\Longrightarrow$ use low frequency ordering.

From (3)

$$\gamma_g P_1 \sim u_1 \frac{P_0}{L}$$

Comparing LHS and P_1 term in (1)

$$\rho_0 \gamma_g \cancel{u_1}, \quad \frac{1}{L} \frac{\cancel{u_1} P_0}{L \gamma_g}$$

$$\gamma_g^2, \quad \gamma_s^2$$

To lowest order

$$P_1 + \frac{B_0 B_1}{4\pi} \simeq 0$$

Add. $\frac{B_0}{4\pi}$ (2) + (3) $\qquad$ $-\rho_0 g$ equilibrium

small

$$\frac{\partial}{\partial t}\cancel{\left(P_1 + \frac{B_0 B_1}{4\pi}\right)} + \overbrace{u_{y1}\left(P_0' + \left(\frac{B_0^2}{8\pi}\right)'\right)}$$

$$+ \nabla \cdot \underset{\sim}{u}_1 \left(\frac{B_0^2}{4\pi} + \Gamma P_0\right) = 0$$

$$\nabla \cdot \underset{\sim}{u}_1 = \frac{u_{y1} \rho_0 g}{\frac{B_0^2}{4\pi} + \Gamma P_0} \equiv \frac{u_{y1}\, g}{\cancel{C_A^2 + C_s^2}\ C_f^2}$$

$$\sim \frac{u_{y1}}{L} \frac{\gamma_g^2}{\gamma_f^2} << \frac{u_{y1}}{L}$$

$$\nabla \cdot \underset{\sim}{u}_1 << \frac{u_{y1}}{L} \Rightarrow \text{nearly incompressible}$$

Must annihilate pressure term in (1)
Operate with $\hat{z} \cdot \nabla \times$ (1)

$$\hat{z} \cdot \nabla \times \left(\rho_0 \frac{\partial}{\partial t} \underset{\sim}{u}_1\right) = -\left(\nabla \rho_1 \times \hat{y}\right) \cdot \hat{z}\, g$$

$$= -g \frac{\partial}{\partial x} \rho_1$$

From (4) with $\nabla \cdot \underset{\sim}{u}_1 \simeq 0$

$$\frac{\partial \rho_1}{\partial t} + u_{1y} \rho_0' = 0$$

Note that P_1, B_1 drop out.

$$\cancel{+}\cancel{i}\omega \cancel{i}k_x \rho_0 \frac{\partial}{\partial t} \hat{u}_y + \frac{\partial}{\partial y} \rho_0 (\pm i\omega) \hat{u}_x$$

$$= -g\, ik\, \hat{\rho} = -g \frac{\cancel{i}k\, \rho_0'}{\cancel{i}\omega} \hat{u}_y$$

$$-i\omega \hat{\rho} + \hat{u}_y \rho_0' = 0$$

$$\omega^2 \left(\rho_0 k \hat{u}_y + i \frac{\partial}{\partial y} \rho_0 \hat{u}_x \right) = -g k \frac{\rho_0'}{\cancel{\rho_0}} \hat{u}_y$$

$$ik \hat{u}_x + \frac{\partial}{\partial y} \hat{u}_y = 0 \implies u_x = -\frac{1}{ik} \frac{\partial}{\partial y} \hat{u}_y$$

(5) $$\boxed{\omega^2 \left(\frac{\partial}{\partial y} \rho_0 \frac{\partial}{\partial y} \hat{u}_y - k^2 \hat{u}_y \rho_0 \right) = g k^2 \rho_0' \hat{u}_y}$$

Solve equation in two limits

$\implies$ first take $kL \gg 1$

$$\implies \frac{\partial}{\partial y} \ll k$$

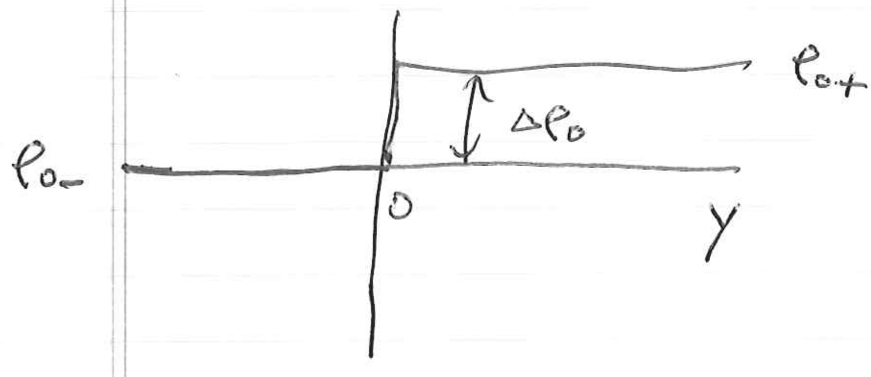
$$\omega^2 = -g \frac{\rho_0'}{\rho_0} \equiv -\gamma_g^2 \implies \gamma^2 = \gamma_g^2$$

$$\gamma_g^2 = g \frac{\rho_0'}{\rho_0} > 0 \quad \text{for } \rho_0' \text{ positive}$$

light fluid supporting heavy fluid.

Heavy fluid falling in gravitational field releases energy.

Sharp boundary limit, $kL << 1$



away from boundary

$$\left(\frac{\partial^2}{\partial y^2} - k^2\right)\hat{u}_y = 0 \quad \Rightarrow \quad \hat{u}_y = e^{-ky} \quad y>0$$

$$= e^{ky} \quad y<0$$

near boundary

$$\frac{\partial}{\partial y} \gg k$$

$$\omega^2 \frac{\partial}{\partial y}\rho_0 \frac{\partial}{\partial y}\hat{u}_y = g k^2 \rho_0' \hat{u}_y$$

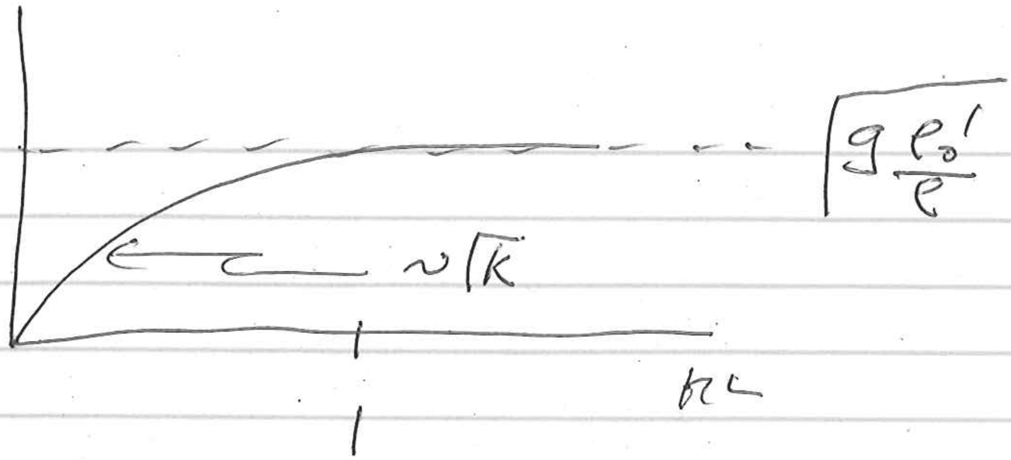
slope of $\hat{u}_y$ undergoes jump. Magnitude of u_y no jump

$$\omega^2 \rho_0 \frac{\partial}{\partial y}\hat{u}_y \Big|_{-\epsilon}^{\epsilon} = g k^2 \Delta\rho_0 \hat{u}_y(0)$$

$$\omega^2\left(-k\rho_0^+ - k\rho_0^-\right)\cancel{\hat{u}_y(0)} = g k^2 \Delta\rho_0 \cancel{u_y(0)}$$

$$\omega^2 = -gk\frac{\Delta\rho_0}{(\rho_0^+ + \rho_0^-)}$$

$$\boxed{\gamma^2 = gk\frac{\Delta\rho}{\rho_0^+ + \rho_0^-}}$$



Notice that the magnetic field is not able to stop the growth of the instability. B_0 is simply convected with the flow.

Curvature driven interchange instability

Proceed as earlier. Again have $\nabla \cdot \underset{\sim}{u}_1 \approx 0$.

~~Opine~~

To lowest order have

$$\underset{\sim}{J}_\perp = c\,\frac{\underset{\sim}{B} \times \nabla P}{B^2}$$

$$0 = -\nabla P + \left(\frac{\underset{\sim}{J} \times \underset{\sim}{B}}{c}\right)$$

Take $\underset{\sim}{b} \cdot \nabla \times$ (momentum eqn)

$$\underset{\sim}{b} \cdot \nabla \times \rho_0 \frac{\partial}{\partial t} \underset{\sim}{u} = \left[\frac{1}{c}\, \underset{\sim}{b} \cdot \nabla \times (\underset{\sim}{J} \times \underset{\sim}{B})\right]$$

$$= \frac{1}{c}\, \underset{\sim}{b} \cdot \left[\underset{\sim}{B} \cdot \nabla \underset{\sim}{J} - \underset{\sim}{J} \cdot \nabla \underset{\sim}{B}\right]$$

$$= \frac{1}{c}\left[\underset{\sim}{B} \cdot \nabla J_\parallel - \underset{\sim}{J} \cdot \underset{\sim}{\kappa}\, B - \underbrace{\left(\underset{\sim}{J} \cdot \nabla \frac{B^2}{2}\right)}_{J_\parallel \frac{\underset{\sim}{B} \cdot \nabla B}{B} + J_\perp \cdot \nabla \frac{B^2}{2}} \frac{1}{B}\right]$$

$$= \frac{1}{c}\left[B\, \underset{\sim}{B} \cdot \nabla \left(\frac{J_\parallel}{B}\right) - B\, \underset{\sim}{J} \cdot \underset{\sim}{\kappa} - \underset{\sim}{J}_\perp \cdot \nabla \frac{B^2}{2}\right]$$

$$= \frac{B}{\cancel{c}}\left[\frac{\underset{\sim}{B} \cdot \nabla}{c}\left(\frac{J_\parallel}{B}\right) - \frac{\underset{\sim}{b} \times \nabla P \cdot \underset{\sim}{\kappa}}{B} - \frac{b \times \nabla P}{B} \cdot \underbrace{\nabla \frac{B^2}{2}}_{\underset{\sim}{\kappa}}\right]$$

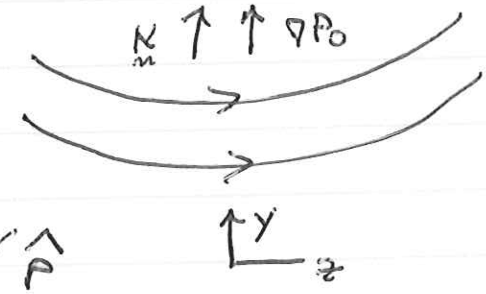
$$0 = -\nabla_\perp P - \frac{1}{8\pi} \nabla_\perp B^2 + \frac{1}{4\pi} B^2 \underset{\sim}{\kappa}$$

$$= B\left[\frac{\underset{\sim}{B} \cdot \nabla}{c}\left(\frac{J_\parallel}{B}\right) + 2\,\frac{\underset{\sim}{b} \times \underset{\sim}{\kappa} \cdot \nabla P}{B}\right]$$

Consider bent field lines with no twist.

$$\Rightarrow J_{\parallel} = 0$$

$$b \cdot \nabla \times \rho_0 \frac{\partial}{\partial t} \underset{\sim}{u}_1 = 2 \underset{\sim}{b} \times \underset{\sim}{\kappa} \cdot \nabla P_1$$



$$\frac{\partial}{\partial t} P_1 + \underset{\sim}{u}_{y1} \frac{\partial}{\partial y} P_0 = 0$$

$$-i\omega \rho_0 \left(b \cdot \nabla \times \hat{\underset{\sim}{u}}_1 \right) = 2 \underset{\sim}{b} \times \underset{\sim}{\kappa} \cdot \underset{\sim}{k} \, \hat{P}$$

$$-i\omega \hat{P} + \hat{u}_y P_0' = 0 \qquad i k \hat{u}_x + \frac{\partial}{\partial y} \hat{u}_y = 0$$

$$\underbrace{i k \hat{u}_y - \frac{\partial}{\partial y} \hat{u}_x}$$

$$i k \hat{u}_y + \frac{\partial}{\partial y} \frac{\hat{u}_y'}{ik}$$

$$\frac{\hat{u}_y'' - k^2 \hat{u}_y}{ik}$$

$$-\omega \rho_0 \frac{\hat{u}_y'' - k^2 \hat{u}_y}{ik} = 2 \underset{\sim}{b} \times \underset{\sim}{\kappa} \cdot \underset{\sim}{k} \frac{\hat{u}_y P_0'}{\omega}$$

$$\omega^2 \rho_0 \left(\hat{u}_y'' - k^2 \hat{u}_y \right) = -2k \, \underset{\sim}{b} \times \underset{\sim}{\kappa} \cdot \underset{\sim}{k} \, P_0' \hat{u}_y$$

$\Rightarrow$ local theory

$$\omega^2 = 2 \frac{1}{k} \underset{\sim}{b} \times \underset{\sim}{\kappa} \cdot \underset{\sim}{k} \frac{P_0'}{\rho_0}$$

$\gamma^2 \sim \frac{1}{R_c} \frac{C_s^2}{L}$

$$\boxed{\omega^2 = -2 \frac{\kappa_y P_0'}{\rho_0}}$$

unstable for ~~$\kappa_y P_0$~~ $\underset{\sim}{\kappa} \cdot \nabla P_0 > 0$

$\Rightarrow g \sim \frac{C_s^2}{R_c} \Rightarrow$ centrifugal force