Ideal MHD waves

Examine the "linear response" of the plasma described by the MHD equations.

Consider a homogeneous equilibrium with $\underset{\sim}{B_0} = B_0 \hat{z}$, P_0, ρ_0, $\underset{\sim}{u_0} = 0$.

linearized equations

Let $P_1(\underset{\sim}{x}, t) = \mathrm{Re}\left(\hat{P}\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{x} - i\omega t}\right)$

etc.

(1) $$-\cancel{i}\omega \rho_0 \hat{\underset{\sim}{u}} = -\cancel{i}\underset{\sim}{k}\left(\hat{P} + \frac{\underset{\sim}{B_0}\cdot\hat{\underset{\sim}{B}}}{4\pi}\right) + \frac{1}{4\pi}\underset{\sim}{B_0}\cdot\cancel{i}\underset{\sim}{k}\,\hat{\underset{\sim}{B}}$$

$$+\cancel{i}\omega \hat{B} + \cancel{i}\underset{\sim}{k}\times\left(\hat{\underset{\sim}{u}}\times\underset{\sim}{B_0}\right) = 0$$

(2) $$\omega \hat{\underset{\sim}{B}} = -\underset{\sim}{k}\cdot\underset{\sim}{B_0}\,\hat{\underset{\sim}{u}} + \underset{\sim}{B_0}\,\underset{\sim}{k}\cdot\hat{\underset{\sim}{u}}$$ (2') $\underset{\sim}{k}\cdot\hat{\underset{\sim}{B}} = 0$

$$\left.\begin{array}{l} -\cancel{i}\omega\hat{P} + \Gamma P_0\,\cancel{i}\underset{\sim}{k}\cdot\hat{\underset{\sim}{u}} = 0 \\ -\cancel{i}\omega\hat{\rho} + \rho_0\,\cancel{i}\underset{\sim}{k}\cdot\hat{\underset{\sim}{u}} = 0 \end{array}\right\}$$ ~~ω(P̂ − ρ̂~~

$$-\omega\hat{P} + \Gamma P_0 \frac{\omega\hat{\rho}}{\rho_0} = 0$$

(3) $$\omega\left(\hat{P} - c_s^2\hat{\rho}\right) = 0 \qquad c_s^2 = \frac{\Gamma P_0}{\rho_0}$$

(4) $$\omega\hat{\rho} = \rho_0\,\underset{\sim}{k}\cdot\hat{\underset{\sim}{u}}$$

There are 8 evolution equations with one constraint $\Rightarrow \underset{\sim}{k} \cdot \hat{\underset{\sim}{B}} = 0$

This implies will have seven solutions for a given $\underset{\sim}{k}$.

To simplify the discussion we will first consider two special cases:

$$k_{\parallel} = 0, \ \underset{\sim}{k}_{\perp} \neq 0 \quad \text{and} \quad k_{\parallel} \neq 0, \ \underset{\sim}{k}_{\perp} = 0$$

<u>Case I</u> $k_{\parallel} = 0, \ \underset{\sim}{k}_{\perp} \neq 0$

Dot (1) with $\underset{\sim}{k}$

(5) $$\omega \rho_0 \, \underset{\sim}{k} \cdot \hat{\underset{\sim}{u}} = k_{\perp}^2 \left(\hat{p} + \frac{B_0 \cdot \hat{B}}{4\pi} \right)$$

Take $\underset{\sim}{k} \times$ (1)

(6) $$\omega \, \underset{\sim}{k} \times \hat{\underset{\sim}{u}} = 0$$

Take $B_0 \times$ (2) — Take $\underset{\sim}{B}_0 \cdot$ (2)

(7) $$\omega \, \underset{\sim}{B}_0 \times \hat{\underset{\sim}{B}} = 0$$ (7') $$\underset{\sim}{k} \cdot \hat{\underset{\sim}{B}} = 0$$ (7'') $$\omega \, \underset{\sim}{B}_0 \cdot \hat{\underset{\sim}{B}} = B_0^2 \, \underset{\sim}{k} \cdot \hat{\underset{\sim}{u}}$$

(8) $$\omega \left(\hat{p} - c_s^2 \hat{\rho} \right) = 0$$

(9) $$\omega \hat{\rho} = \rho_0 \, \underset{\sim}{k} \cdot \hat{\underset{\sim}{u}}$$

Have only one mode with non zero frequency $\Rightarrow$ driven by $\mathbf{k}\cdot\hat{\mathbf{u}} \neq 0$

$$\omega \rho_0\, \mathbf{k}\cdot\hat{\mathbf{u}} = k_\perp^2\left(\frac{c_s^2 \rho_0}{\omega} + \frac{1}{4\pi}\frac{B_0^2}{\omega}\right)\mathbf{k}\cdot\hat{\mathbf{u}}$$

$$\boxed{\begin{array}{c}\omega^2 = k_\perp^2\left(c_s^2 + c_A^2\right) \\ \mathbf{k}\cdot\hat{\mathbf{u}} \neq 0,\ \hat{P} \neq 0,\ \hat{\rho} \neq 0,\ \mathbf{B}_0\cdot\hat{\mathbf{B}} \neq 0\end{array}}$$

$$c_A^2 = \frac{B_0^2}{4\pi\rho_0}$$

Note: 2 modes $\pm k_\perp\left(c_s^2 + c_A^2\right)^{\frac{1}{2}}$

$\Rightarrow$ magneto sonic wave

Since $\mathbf{k}\cdot\hat{\mathbf{u}} \neq 0$, this is a compressional mode like a sound wave but the frequency is increased by the compression of the magnetic field.

$\Rightarrow$ also known as the "fast" wave.

$k_\parallel = 0$, $k_\perp \neq 0$ modes with $\omega = 0$

Take $\underset{\sim}{k}_\perp = k_\perp \hat{x}$

ⓐ Entropy mode — ⑧

$\hat{p} = 0$ so no $\underset{\sim}{k} \cdot \hat{\underset{\sim}{u}}$

$\left\{ \hat{\rho} \neq 0 \text{ but } \hat{p} = 0 \right.$

$$\Rightarrow \hat{\rho} T_0 + \rho_0 \hat{T} = 0$$

$$\hat{T} = -\frac{\hat{\rho}}{\rho_0} T_0 \neq 0$$

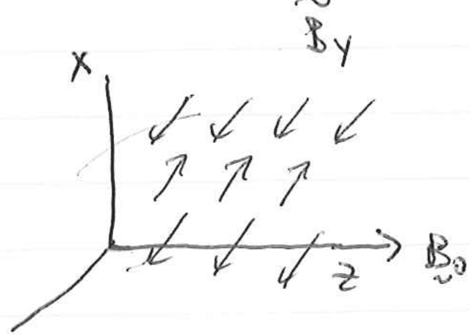
$\frac{p}{\rho^\gamma}$ = entropy

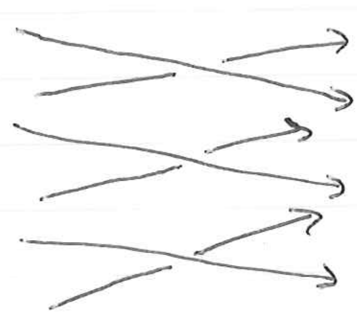
note entropy perturbation $\sim -\hat{\rho} \neq 0$

ⓑ Magnetic Shear mode — ⑦

$\tilde{B}_y \neq 0$, $\underset{\sim}{B_0} \times \hat{\underset{\sim}{B}} \neq 0$

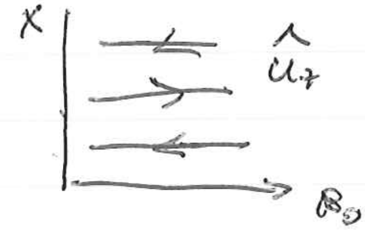
magnetic field twists as a function of x.





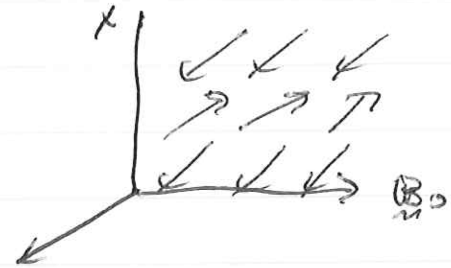
(c) Parallel flow mode — (6)

$\hat{u}_z \neq 0$, $\underset{\sim}{k} \times \hat{\underset{\sim}{u}} \neq 0$



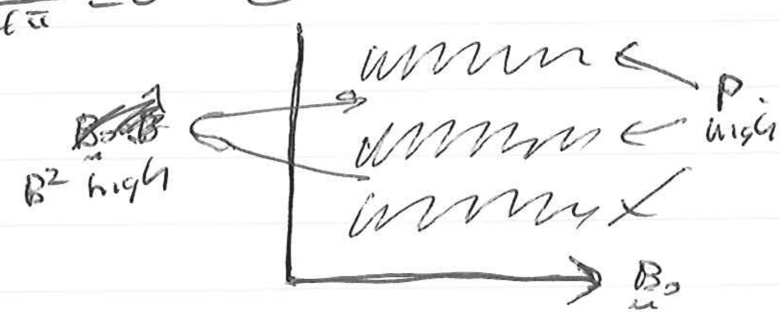
(d) "convective cell" (6)

$\hat{u}_y \neq 0$, $\underset{\sim}{k} \times \hat{\underset{\sim}{u}} \neq 0$



(e) "Diamagnetic mode" — (8)

$\hat{P} \neq 0$, $\hat{P} + \frac{\underset{\sim}{B}_0 \cdot \hat{\underset{\sim}{B}}}{4\pi} = 0$ — (5)



Case II $k_\parallel \neq 0$, $k_\perp = 0$

take $\mathbf{k}\cdot$ (1) since $\mathbf{k}\cdot\hat{\mathbf{B}} = 0$

(10) $\omega\rho_0\,\mathbf{k}\cdot\hat{\mathbf{u}} = k_\parallel^2\left(\hat{p} + \cancelto{0}{\frac{\mathbf{B}_0\cdot\hat{\mathbf{B}}}{4\pi}}\right)$

take $\mathbf{k}\times$ (1)

(11) $\omega\rho_0\,\mathbf{k}\times\hat{\mathbf{u}} = -\frac{1}{4\pi}\, k_\parallel B_0\,\mathbf{k}\times\hat{\mathbf{B}}$

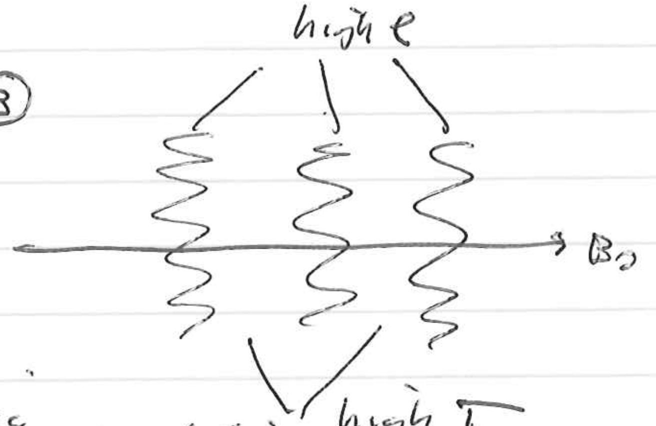
take $\mathbf{k}\times$ (2)

(12) $\omega\,\mathbf{k}\times\hat{\mathbf{B}} = -k_\parallel B_0\,\mathbf{k}\times\hat{\mathbf{u}}$

(13) $\omega\left(\hat{p} - c_s^2\hat{\rho}\right) = 0$

(14) $\omega\hat{\rho} = \rho_0\,\mathbf{k}\cdot\hat{\mathbf{u}}$

(a) Entropy mode — (13)



$\hat{\rho} \neq 0$, $\hat{T} \neq 0$

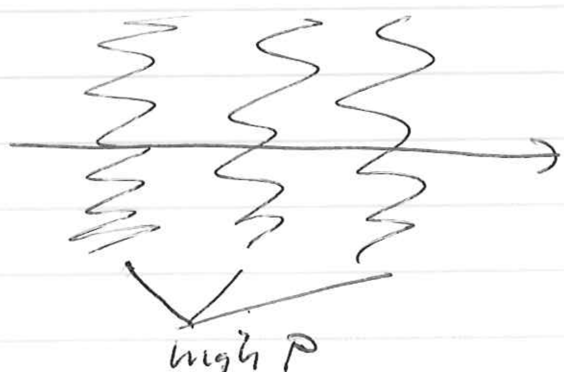
$\hat{p} = 0$ etc.

$\omega = 0$

⇒ high ~~ρ~~ unrealistic

⇒ ∥ thermal conduction

(b) Sound waves — (10), (13), (14)



$\omega\cancel{\rho_0}\,\mathbf{k}\cdot\hat{\mathbf{u}} = k_\parallel^2 c_s^2 \frac{\cancel{\rho_0}}{\omega}\,\mathbf{k}\cdot\hat{\mathbf{u}}$

$\omega^2 = k_\parallel^2 c_s^2$

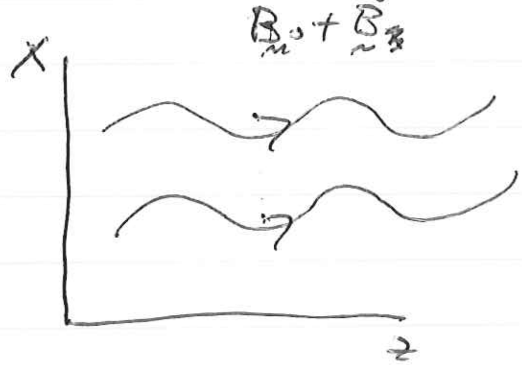
$\mathbf{k}\cdot\hat{\mathbf{u}} \neq 0$, $\hat{p} \neq 0$, $\hat{\rho} \neq 0$

Note: $\omega = \pm k_\parallel c_s$

(c) Alfven waves — (11), (12)

$$\omega \rho_0\, k\times\hat{u} = +\frac{1}{4\pi} k_\| B_0 \frac{1}{\omega}\left(+k_\| B_0\right) k\times\hat{u}$$

$$\omega^2 = k_\|^2 C_A^2$$

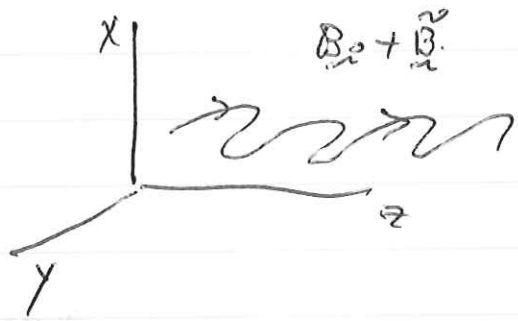


$k\times\hat{u},\ k\times\hat{B} \neq 0$

two polarizations:

$\tilde{B}_X,\ \tilde{u}_X \neq 0$

$\tilde{B}_Y,\ \tilde{u}_Y \neq 0$



Note: 4 modes
$\pm k_\| C_A$ each pol.

<u>Case III</u> General case ~~$k\cdot B_0 \neq 0$~~
$k_\| \neq 0,\ k_\perp \neq 0$

From (3), (4) $\hat{P} = c_s^2 \frac{1}{\omega} \rho_0\, k\cdot\hat{u}$

$k\cdot$(1) $\omega \rho_0\, k\cdot\hat{u} = k^2\left(\hat{P} + \frac{B_0\cdot\hat{B}}{4\pi}\right)$

(15) $$\omega^2 \rho_0\, k\cdot\hat{u} = k^2\left(c_s^2 \rho_0\, k\cdot\hat{u} + \omega\frac{B_0\cdot\hat{B}}{4\pi}\right)$$

~~$k\times$(1)~~

~~$\omega\rho_0\, k\times\hat{u} = -\frac{1}{4\pi} k\cdot B_0\, k\times\hat{B}$~~

$B_0\cdot$(1) $\omega \rho_0\, \hat{u}\cdot B_0 = k\cdot B_0\left(\frac{c_s^2}{\omega} \rho_0\, k\cdot\hat{u} + \omega\cancel{\frac{B_0\cdot\hat{B}}{4\pi}}\right)$

$- \frac{1}{4\pi} k\cdot B_0\, B_0\cdot\hat{B}\,\omega$

(16) $\omega \cancel{\rho_0}\, \hat{u}\cdot B_0 = c_s^2 \cancel{\rho_0}\, k\cdot B_0 \; k\cdot\hat{u}\, \frac{1}{\omega}$

$k\times B_0 \cdot$ (1)

(17) $\omega \rho_0\, \hat{u}\cdot k\times B_0 = -\frac{1}{4\pi}\, k\cdot B_0 \; k\times B_0\cdot\hat{B}$

(18) $B_0\cdot$ (2). $\quad \omega\, B_0\cdot\hat{B} = -k\cdot B_0 \; B_0\cdot\hat{u} + B_0^2\, k\cdot\hat{u}$

$k\times B_0 \cdot$ (2)

(19) $\omega\, k\times B_0\cdot\hat{B} = -k\cdot B_0 \; k\times B_0\cdot\hat{u}$

Using (15), (18), (16)

$$\omega^2\, k\cdot\hat{u} = k^2\left(c_s^2\, k\cdot\hat{u} + \frac{1}{4\pi\rho_0}\left(B_0^2\, k\cdot\hat{u} - k\cdot B_0^{\,2}\, \frac{c_s^2}{\omega^2}\, k\cdot\hat{u}\right)\right)$$

$$\omega^2 = k^2 c_s^2 + k^2 c_A^2 - \frac{c_s^2}{4\pi\rho_0}\frac{(k\cdot B_0)^2}{\omega^2}k^2$$

$$(\omega^2 - k^2c_s^2 - k^2c_A^2)\,\omega^2 + k^2c_s^2\,k_\parallel^2\,c_A^2 = 0$$

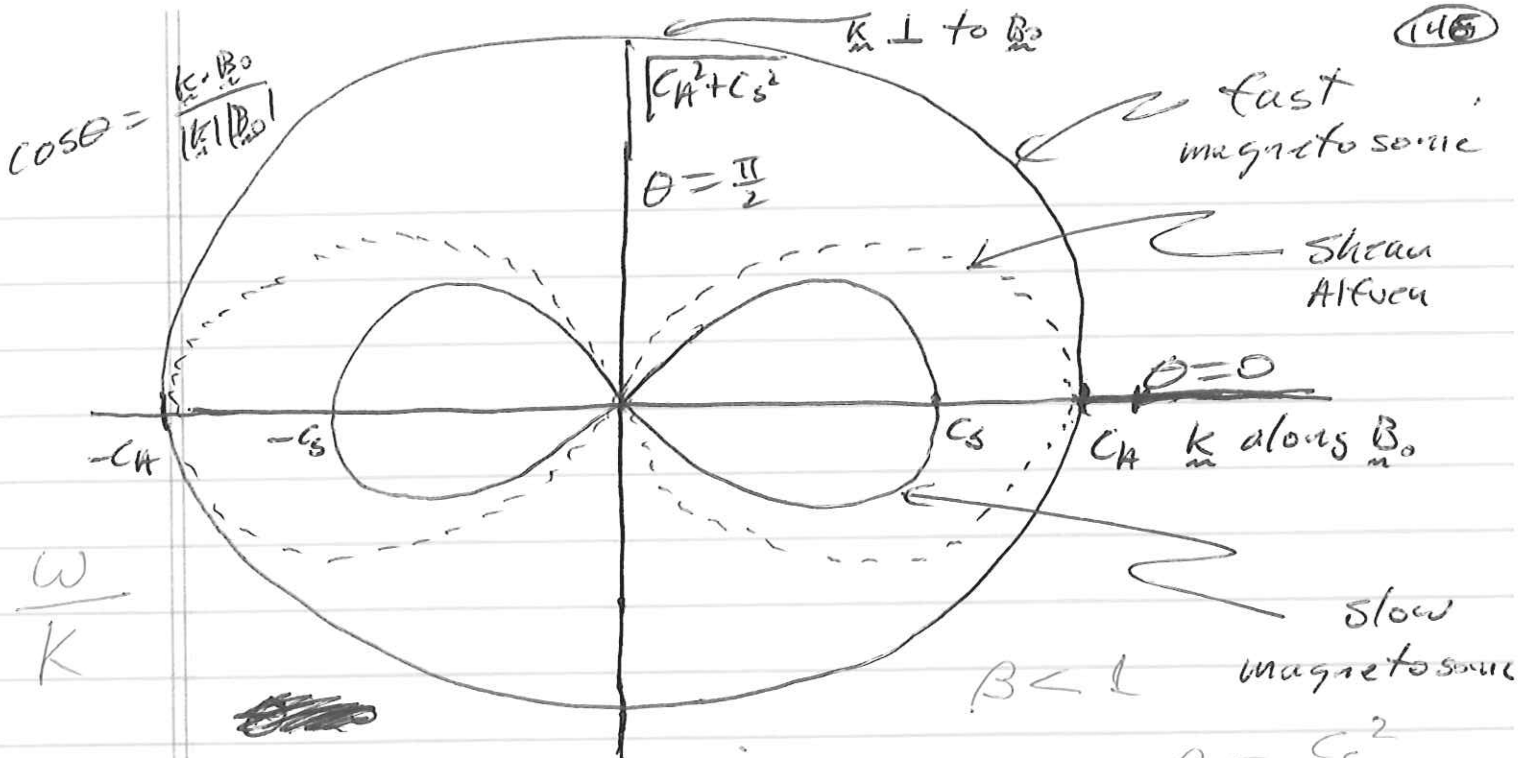
$$c_{ms}^2 = c_s^2 + c_A^2$$ $\quad c_f \equiv c_{ms}$ fast wave

$$(\omega^2 - k^2c_{ms}^2)\,\omega^2 + k^2c_s^2\,k_\parallel^2\,c_A^2$$

$$\boxed{\omega^2 = \frac{k^2c_{ms}^2 \pm \sqrt{k^4c_{ms}^4 - 4\,k^2c_s^2\,k_\parallel^2\,c_A^2}}{2}}$$

Using (17), (19) $\quad \omega^2\rho_0\, \hat{u}\cdot k\times B_0 = +\frac{1}{4\pi}\, k_\parallel^2 B_0^2\,(k\times B_0\cdot\hat{u})$

$$\boxed{\omega^2 = k_\parallel^2 c_A^2}$$ shear Alfven $\hat{p} = 0$, $\hat{B}\cdot B_0 = 0$



parallel propagation $k_\parallel = k$

$$\omega^2 = \frac{k^2 c_{ms}^2 \pm k^2\sqrt{c_{ms}^4 - 4c_s^2 c_A^2}}{2}$$

$$= \frac{k^2(c_A^2+c_s^2) \pm k^2\sqrt{(c_A^2-c_s^2)^2}}{2}$$

$$= k^2c_A^2,\ k^2c_s^2$$

$\perp$ propagation $k_\parallel = 0$

$$\omega^2 = k^2 c_{ms}^2,\ 0$$

~~Strong B_0 limit~~

$\beta \gg 1$