# The MHD (magnetohydrodynamic) equations

Want to develop a set of fluid equations that can be used to model plasma dynamics in a magnetic field. Will use the Boltzmann equation with collision operator and Maxwell's Eqns.

To justify the fluid model, need to assume that the collision rate is high

$\Rightarrow$ high enough so that both species are drifting Maxwellians $\Rightarrow$ same T

~~$\Rightarrow$ don't require equal temperatures~~

$\Rightarrow$ consider for simplicity ~~the~~ the masses to be of the same order so ~~don't distinguish~~ use same ordering for each species.

Assume the following

$$u \sim V_t \sim C_A \sim \text{Alfven speed} \sim \frac{B}{\sqrt{4\pi m n}}$$

$$\omega \sim V_t / L \;, \quad \nu \sim \Omega$$

Define a small parameter

$$\epsilon \equiv \frac{\omega}{\nu} \sim \left(\frac{V_t}{c}\right)^2 \sim \frac{r_L}{L}$$

Boltzmann Eqn for species $\alpha, \beta$

$$\frac{\partial f_\alpha}{\partial t} + \underset{\sim}{v}\cdot\nabla f_\alpha + \left(\frac{q}{m}\right)_\alpha\left(\underset{\sim}{E}+\frac{1}{c}\underset{\sim}{v}\times\underset{\sim}{B}\right)\cdot\frac{\partial}{\partial \underset{\sim}{v}} f_\alpha = C_{\alpha\alpha}+C_{\alpha\beta}$$

$$\omega \qquad \omega \qquad \frac{q}{m}\frac{E}{v_t} \qquad \frac{q}{m}\frac{1}{c}B \qquad \nu \qquad \nu$$

$$\frac{\omega}{\cancel{\ }} \qquad \frac{\omega}{\cancel{\ }} \qquad \frac{qB}{mc}\left(\frac{cE}{Bv_t}\right) \qquad \Omega \qquad \nu \qquad \nu$$

$$\epsilon \qquad \epsilon \qquad 1 \qquad 1 \qquad 1 \qquad 1$$

Gauss' Law

$$\nabla\cdot\underset{\sim}{E} = 4\pi e\left(\int d\underset{\sim}{v}\, f_i - \int dv\, f_e\right)$$

$$\frac{B}{c}\frac{cE}{BL} \qquad 4\pi e n_i \qquad 4\pi e n_e$$

get rid of n through $c_A^2 = B^2/4\pi m n$

$$\frac{v_t \, B^2 mc}{m 4\pi L \, n_i q c^2 B}, \quad 1, \quad 1$$

$$\frac{v_t^3}{c^2 L\Omega}, \quad 1, \quad 1$$

$$\left(\frac{v_t}{c}\right)^2\frac{r_L}{L}, \quad 1, \quad 1$$

$$\epsilon^2, \quad 1, \quad 1 \quad \Rightarrow \text{quasi-neutrality}$$

~~Ea~~ Faradays Law

~~Ea~~

$$\frac{1}{c}\frac{\partial \underset{\sim}{B}}{\partial t} + \nabla \times \underset{\sim}{E} = 0$$

$$\frac{\omega}{\cancel{c}}\cancel{B}\ ,\ \frac{E}{L}\frac{c}{B}\cancel{B}$$

$$\omega\ ,\ \frac{V_t}{L}$$

$$1\ ,\ 1$$

Ampere's Law

$$\nabla \times \underset{\sim}{B} = \frac{1}{c}\frac{\partial \underset{\sim}{E}}{\partial t} + \frac{4\pi}{c} e \int d\underset{\sim}{v}\, \underset{\sim}{v} \left(f_i - f_e\right)$$

$$\frac{B}{L}\ ,\ \frac{\omega}{c}E\ ,\ \frac{4\pi e}{c} n_i v_t\ ,\ \frac{4\pi e}{c}\frac{\cancel{n_e} v_t m}{\cancel{B^2} C_A^2} \quad \frac{B^2}{m}$$

$$\frac{\cancel{B}}{L}\frac{m}{B}\frac{c}{e}\ ,\ \frac{\omega}{c^2} v_t \frac{m}{B}\frac{c}{e}\ ,\ \frac{1}{v_t}\ ,\ \frac{1}{v_t}$$

$$\frac{r_L}{L}\ ,\ \frac{v_t^2}{c^2}\frac{\omega}{\Omega}\ ,\ 1\ ,\ 1$$

$$\epsilon\ ,\ \epsilon^2\ ,\ 1\ ,\ 1$$

<u>Lowest order</u>

① $\left(\frac{q}{m}\right)_\alpha \left(\mathbf{E}_0 + \frac{1}{c}\mathbf{v}\times\mathbf{B}_0\right)\cdot\frac{\partial}{\partial \mathbf{v}} f_{\alpha 0} = C_{\alpha\alpha 0} + C_{\alpha\beta 0}$

② $0 = n_i^0 - n_e^0$ ⟸ quasi-neutrality

③ $\boxed{\frac{1}{c}\frac{\partial \mathbf{B}_0}{\partial t} + \nabla\times\mathbf{E}_0 = 0}$

④ $0 = (n_i \mathbf{u}_i)_0 - (n_e \mathbf{u}_e)_0 \Rightarrow \mathbf{J}_0 = 0$

From ① ⟹ Maxwellian with $n_i = n_e$, $\mathbf{u}_0^i = \mathbf{u}_0^e = \mathbf{u}_0$ and $T_e = T_i$

⟹ eliminates $C_{\alpha\alpha 0}$, $C_{\alpha\beta 0}$

$$\left(\mathbf{E}_0 + \frac{1}{c}\mathbf{v}\times\mathbf{B}_0\right)\cdot(\mathbf{v}-\mathbf{u}_0) = 0$$

$$\mathbf{E}_0\cdot(\mathbf{v}-\mathbf{u}_0) - \frac{1}{c}\mathbf{u}_0\cdot\mathbf{v}\times\mathbf{B}_0 = 0$$

$$\left(\mathbf{E}_0 + \frac{1}{c}\mathbf{u}_0\times\mathbf{B}_0\right)\cdot\mathbf{v} = \mathbf{E}_0\cdot\mathbf{u}_0$$

Valid for any $\mathbf{v}$ so $\mathbf{E}_0 + \frac{1}{c}\mathbf{u}_0\times\mathbf{B}_0 = 0$

⟹ $\mathbf{u}_{\perp 0}$ is the $\mathbf{E}\times\mathbf{B}_0$ drift.

⟹ $E_{0\parallel} u_{0\parallel} = 0$

⟹ $E_{0\parallel} = 0$

$$\boxed{\mathbf{E}_0 = -\frac{1}{c}\mathbf{u}_0\times\mathbf{B}_0}$$

⇓ $\mathbf{E}_0 = 0$ in plasma frame ⟹ good conductor

First order in $\epsilon$

$$\frac{\partial f_{\alpha 0}}{\partial t} + \underset{\sim}{v} \cdot \nabla f_{\alpha 0} + \left(\frac{q}{m}\right)_\alpha \left[\left(\underset{\sim}{E} + \frac{1}{c} \underset{\sim}{v} \times \underset{\sim}{B}\right) \cdot \frac{\partial}{\partial \underset{\sim}{v}} f_\alpha\right]_1$$

$$= \quad C_{\alpha\alpha 1} + C_{\alpha\beta 1}$$

$$0 = n_{i1} - n_{e1}$$

$$\frac{1}{c}\frac{\partial \underset{\sim}{B}_1}{\partial t} + \nabla \times \underset{\sim}{E}_1 = 0$$

$$\nabla \times \underset{\sim}{B}_0 = \frac{4\pi}{c} e \left[ (n_i u_i)_1 - (n_e \underset{\sim}{u}_e)_1 \right]$$

Recall:

$$\int d\underset{\sim}{v}\, C_{\alpha\alpha} \begin{pmatrix} 1 \\ m_\alpha \underset{\sim}{v} \\ \frac{1}{2} m_\alpha v^2 \end{pmatrix} = 0$$

$\Rightarrow$ self-collisions don't change #, momentum or energy

$$\int dv\, C_{\alpha\beta} \begin{pmatrix} 1 \\ m_\alpha \underset{\sim}{v} \\ \frac{1}{2} m_\alpha v^2 \end{pmatrix} = \begin{matrix} 0 \\ -R_{\alpha\beta} \\ -\left(\frac{\partial W}{\partial t}\right)_{\alpha\beta} \end{matrix}$$

$$R_{\alpha\beta} + R_{\beta\alpha} = 0$$

$$\left(\frac{\partial W}{\partial t}\right)_{\alpha\beta} + \left(\frac{\partial W}{\partial t}\right)_{\beta\alpha} = 0$$

Can eliminate collision operators by integrating over $\underset{\sim}{v}$

$$\boxed{\frac{\partial}{\partial t} n + \nabla \cdot n \underset{\sim}{u} = 0}$$

suppress subscript o.

~~Integrate over $\int d\underset{\sim}{v}\, \underset{\sim}{v}$~~

Operate with $\int d\underset{\sim}{v}\, \underset{\sim}{v}$

$$n m_\alpha \frac{d}{dt} \underset{\sim}{u} = -\nabla P_\alpha + n q_\alpha \left( \underset{\sim}{E}_1 + \frac{1}{c} \underset{\sim}{u} \times \underset{\sim}{B}_1 \right)$$

$$+ q_\alpha \left( n_{\alpha 1} \underset{\sim}{E} + \frac{1}{c} (n \underset{\sim}{u})_{\alpha 1} \times \underset{\sim}{B} \right) + \underset{\sim}{R}_{\alpha\beta}$$

$\Rightarrow$ sum over $\alpha, \beta$ and use $n_{e1} = n_{i1}$

$\Rightarrow$ define $\rho = (m_\alpha + m_\beta) n$ , $P_\alpha + P_\beta = P$

$$\boxed{\rho \frac{d}{dt} \underset{\sim}{u} = -\nabla P + \frac{1}{c} \underset{\sim}{J}_1 \times \underset{\sim}{B}}$$

$$\underset{\sim}{J}_1 = e (n \underset{\sim}{u})_{i1} - e (n \underset{\sim}{u})_{e1}$$

~~Note~~

Operate with $\int d\underset{\sim}{v}\, v^2$

$$\frac{\partial}{\partial t} \left( \frac{1}{2} n m_\alpha u^2 + \frac{3}{2} P_\alpha \right) + \nabla \cdot \left( \frac{1}{2} n m_\alpha u^2 + \frac{3}{2} P_\alpha \right) \underset{\sim}{u}$$

$$+ \nabla \cdot \left( P_\alpha \underset{\sim}{u} \right) = q_\alpha \left[ n \underset{\sim}{u} \cdot \underset{\sim}{E}_1 + (n u)_{1\alpha} \cdot \underset{\sim}{E} \right] - \left( \frac{\partial W}{\partial t} \right)_{\alpha\beta}$$

Sum over species

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \frac{3}{2}P\right) + \nabla\cdot\left(\frac{1}{2}\rho u^2 + \frac{3}{2}P\right)\underline{u} + \nabla\cdot(P\underline{u})$$

$$= \underline{J}_1\cdot\underline{E} = -\underline{J}_1\cdot\underline{u}\times\underline{B}\,\frac{1}{c}$$

$$\nabla\times\underline{B}_0 = \frac{4\pi}{c}\underline{J}_1 \qquad = \underline{u}\cdot\frac{1}{c}\,\underline{J}_1\times\underline{B}$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \frac{3}{2}P\right) + \nabla\cdot\left(\frac{1}{2}\rho u^2 + \frac{3}{2}P\right)\underline{u} + \nabla\cdot(P\underline{u})$$

$$= \underline{u}\cdot\nabla P + \underbrace{\rho\,\underline{u}\cdot\frac{d}{dt}\underline{u}}_{\rho\frac{d}{dt}\frac{u^2}{2}}$$

$$\frac{1}{2}u^2\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial t}\left(\frac{3}{2}P\right) + \underline{u}\cdot\nabla\left(\frac{1}{2}\rho u^2 + \frac{3}{2}P\right)$$

$$+\left(\frac{1}{2}\rho u^2 + \frac{3}{2}P\right)\nabla\cdot\underline{u} + \cancel{\underline{u}\cdot\nabla P} + P\,\nabla\cdot\underline{u}$$

$$= \cancel{\underline{u}\cdot\nabla P} + \rho\,\underline{u}\cdot\nabla\frac{u^2}{2}$$

$$-\frac{1}{2}u^2\left(+\cancel{\rho\nabla\cdot\underline{u}} + \cancel{\underline{u}\cdot\nabla\rho}\right) + \frac{\partial}{\partial t}\left(\frac{3}{2}P\right) + \cancel{\frac{1}{2}u^2\,\underline{u}\cdot\nabla\rho}$$

$$+\frac{3}{2}\,u\cdot\nabla P + \left(\cancel{\frac{1}{2}\rho u^2} + \frac{5}{2}P\right)\nabla\cdot\underline{u} = 0$$

$$\cancel{\frac{\partial P}{\partial t}} + \frac{d}{dt}P + \frac{5}{3}P\nabla\cdot\underset{\sim}{u} = 0$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\rho\underset{\sim}{u} = 0 \qquad \Rightarrow \rho = (m_i + m_e)n$$

$$\rho\frac{d}{dt}\underset{\sim}{u} = -\nabla P + \frac{1}{c}\underset{\sim}{J}\times\underset{\sim}{B} = -\nabla\left(P + \frac{B^2}{8\pi}\right) + \frac{1}{4\pi}\underset{\sim}{B}\cdot\nabla\underset{\sim}{B}$$

$$\nabla\times\underset{\sim}{B} = \frac{4\pi}{c}\underset{\sim}{J}$$

$$\nabla\cdot\underset{\sim}{B} = 0$$

$$\frac{1}{c}\frac{\partial}{\partial t}\underset{\sim}{B} + \nabla\times\underset{\sim}{E} = 0$$

$$\underset{\sim}{E} = -\frac{1}{c}\underset{\sim}{u}\times\underset{\sim}{B}$$

# Ideal MHD Equilibria

In exploring the stability and dynamics of magnetized plasma, it is useful to start with states that are time stationary so that they are in a state of equilibrium. We will discuss several examples of such states. For simplicity we will limit this discussion to situations without flow although in many cases equilibria with flow can also be constructed. The basic equations are

momentum:

① $0 = -\nabla P + \frac{1}{c}\,\underset{\sim}{J} \times \underset{\sim}{B}$

② $\underset{\sim}{E} = 0$

③ $\nabla \cdot \underset{\sim}{B} = 0$

④ $\nabla \times \underset{\sim}{B} = \frac{4\pi}{c}\,\underset{\sim}{J}$

Alternatively, we can write ① as

$$0 = -\nabla P + \frac{1}{c}\,\frac{c}{4\pi}\underbrace{(\nabla \times \underset{\sim}{B}) \times \underset{\sim}{B}}_{-\nabla \frac{B^2}{2} + \underset{\sim}{B}\cdot\nabla \underset{\sim}{B}}$$

①′ $$0 = -\nabla\left(P + \frac{B^2}{8\pi}\right) + \underset{\sim}{B}\cdot\nabla \underset{\sim}{B}$$

Equ ① requires that

$$\underset{\sim}{B} \cdot \nabla P = 0$$

so that the pressure is constant along $\underset{\sim}{B}$.

$\Rightarrow$ any parallel pressure gradients are relaxed by sound waves.

Equ ④ requires

$$\nabla \cdot \underset{\sim}{J} = 0$$

which implies that charge can not build up. This would produce an $\underset{\sim}{E} \neq 0$ and since

$$\underset{\sim}{E} = -\underset{\sim}{u} \times \underset{\sim}{B} \frac{1}{c}$$

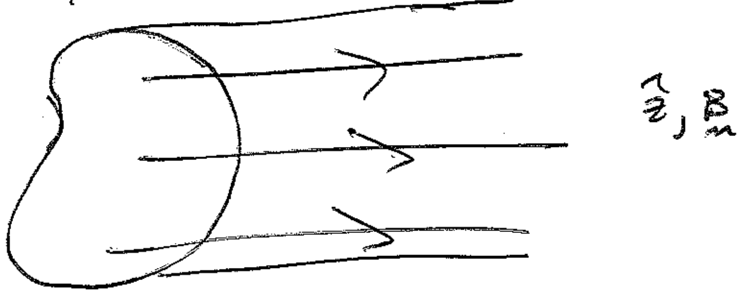
flows would be generated.

<u>Case I</u> $\underset{\sim}{B} \cdot \nabla \underset{\sim}{B} = 0$

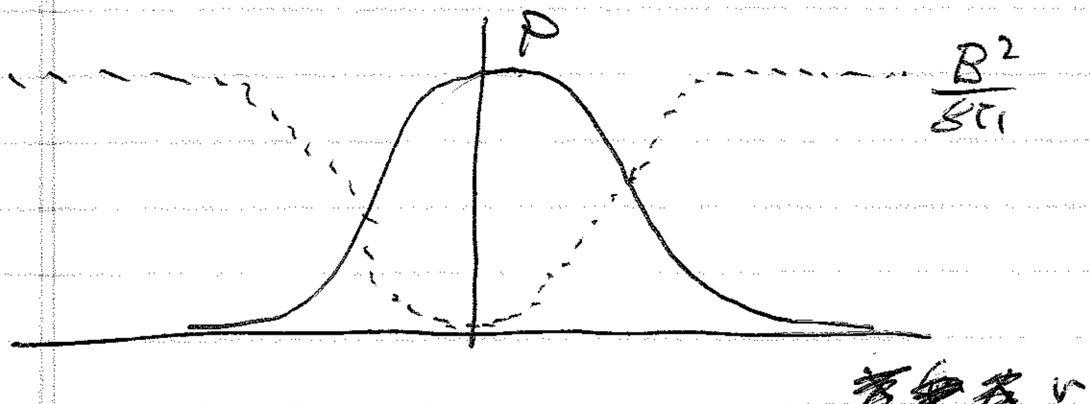
In this case we have straight field lines and

$$\nabla \left( P + \frac{B^2}{8\pi} \right) = 0$$

$\Rightarrow$ the total pressure, magnetic plus plasma must be constant

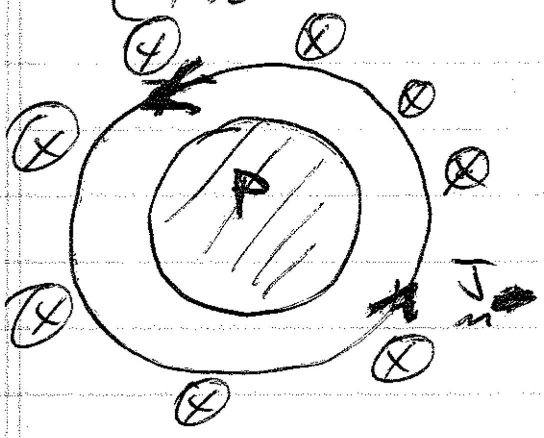


This is not a trivial state, e.g.,




$$\frac{\partial}{\partial r}\left(P + \frac{B^2}{8\pi}\right) = 0$$

Confinement experiments with such equilibria are called $\theta$-pinches (the currents are in the $\theta$ direction)

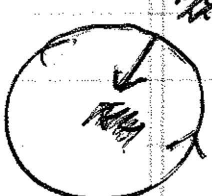



$$-\frac{\partial B_z}{\partial r} = \frac{4\pi}{c} J_\theta$$

<u>Case II</u> $\quad \mathbf{B}\cdot\nabla\mathbf{B} \neq 0$

Consider a pure azimuthal field $B_\theta \hat{\theta}$

$$\mathbf{B}\cdot\nabla\mathbf{B} = B^2 \boldsymbol{\kappa} = \frac{B_\theta}{r}\frac{\partial}{\partial\theta} B_\theta \hat{\theta}$$



$$= -\frac{B_\theta^2}{r}\hat{e}_r$$

$$\Rightarrow R_c = r$$

This ~~force~~ ~~[illegible]~~ results from "magnetic tension", which can balance variations in the total plasma pressure

$$\frac{\partial}{\partial r}\left(P + \frac{B_\theta^2}{8\pi}\right) + \frac{B_\theta^2}{r} = 0$$

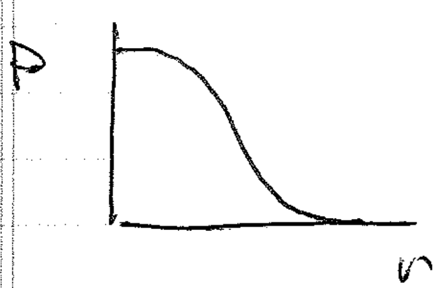
$$\left(\nabla \times B_\theta \hat{\theta}\right)_z = \frac{4\pi}{c} J_z$$

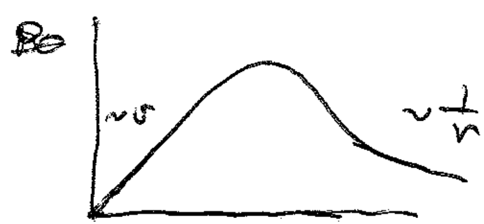
$\Longrightarrow$ current along $\hat{z}$ $\Longrightarrow$ z pinch

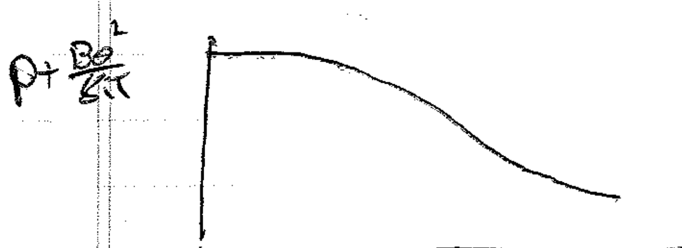
$$\left(\nabla \times r B_\theta \nabla\theta\right)_z = \frac{4\pi}{c} J_z$$

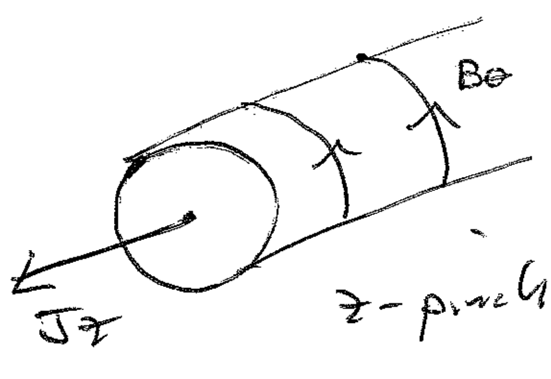
$$\left(\underbrace{\nabla (r B_\theta)}_{\frac{\partial}{\partial r} r B_\theta} \times \underbrace{\nabla\theta}_{\frac{1}{r}}\right)_z = \frac{4\pi}{c} J_z$$

$$\frac{1}{r}\frac{\partial}{\partial r} r B_\theta = \frac{4\pi}{c} J_z$$

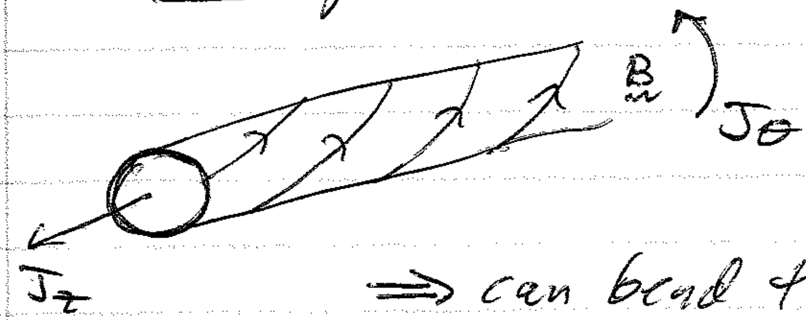


A __screw pinch__ has both $B_\theta, B_z \neq 0$

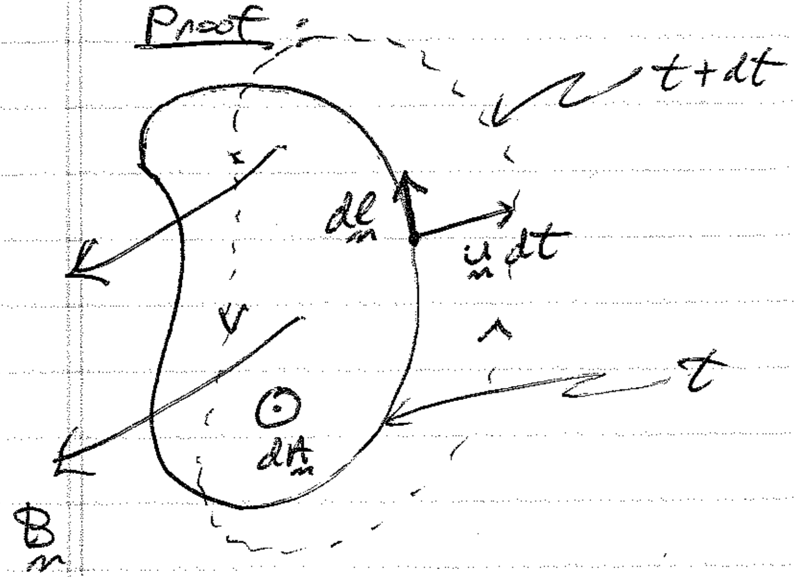



$\Rightarrow$ can bend this into a torus? ~~discuss later~~

## The ~~loop~~ "frozen-in" theorem

In ~~an~~ __ideal__ MHD ~~plasma~~, a plasma moves so that the magnetic flux through any connected fluid element is conserved. $\Rightarrow$ not valid if have dissipation $\Rightarrow$ e.g. resistivity

__Proof:__




$$\Phi = \oint \mathbf{B}\cdot d\mathbf{A}$$

$$\frac{d\Phi}{dt} = \underbrace{\oint \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{A}}_{\text{change due to } \frac{\partial B}{\partial t}} \underbrace{- \frac{1}{dt}\oint d\mathbf{l}\times \mathbf{u}\,dt \cdot \mathbf{B}}_{\text{change due to motion of loop}}$$

$$\dot{\Phi} = \oint \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{A} - \oint \mathbf{u}\times\mathbf{B}\cdot d\mathbf{l}$$

$$= -c \oint \nabla\times\mathbf{E}\cdot d\mathbf{A} - \oint \mathbf{u}\times\mathbf{B}\cdot d\mathbf{l}$$

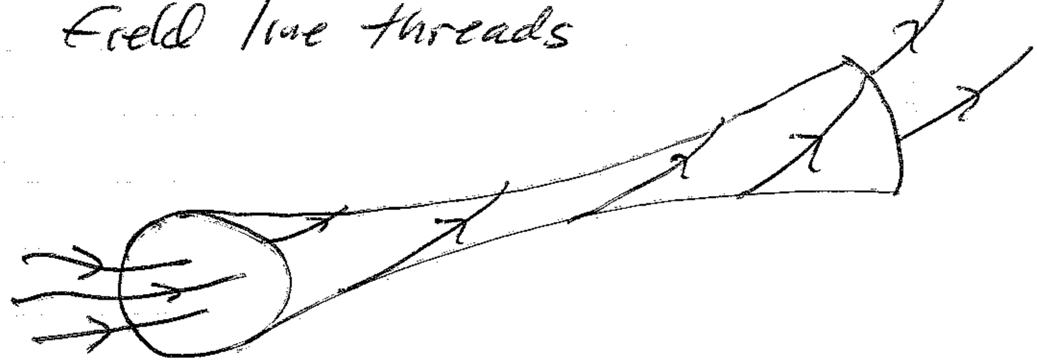
$$= -c \oint \left(\mathbf{E} + \frac{1}{c}\mathbf{u}\times\mathbf{B}\right)\cdot d\mathbf{l}$$

$$= 0 \quad \text{since} \quad \mathbf{E} = -\frac{1}{c}\mathbf{u}\times\mathbf{B}$$

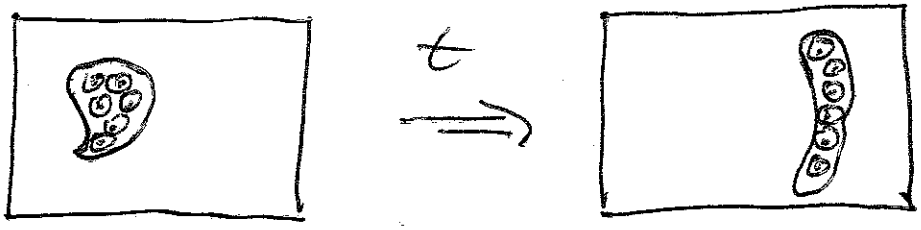
Note that must have $E_{\parallel} = 0$.

~~Define a flux tube ... is~~

A "flux tube" is defined by a bounding surface through which no magnetic field line threads



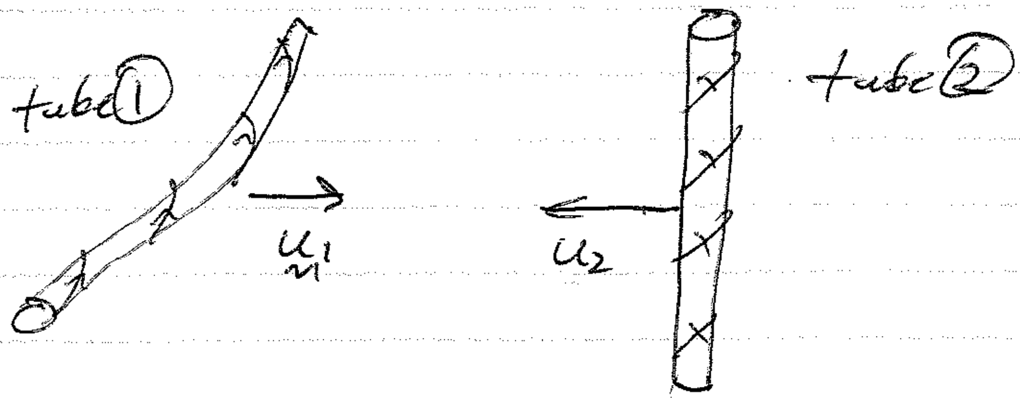
Corollary 1: when a plasma moves, it carries its flux tube with it.

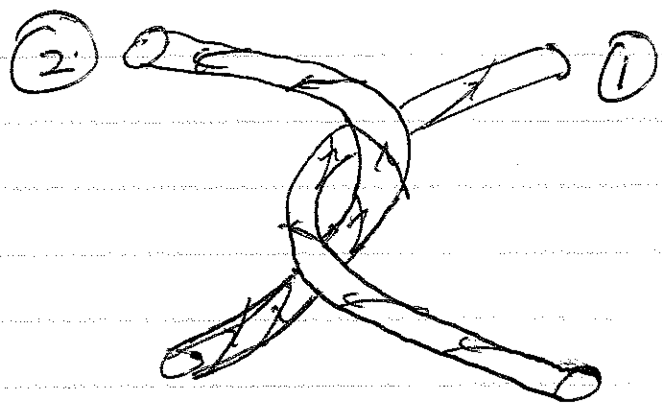


Consider region of localized field and surrounding plasma. As the plasma moves the field lines move with it.

## Corollary 2

Two "crossed" flux tubes can not pass through each other.




$\Rightarrow$ flux tubes do not change topology