Bellan - Fundamentals of Plasma Physics
Ch 3. G and R Ch 3

Particle Orbits in Magnetic Fields

We are now going to start discussing the dynamics of plasmas with embedded magnetic fields. The dynamics are linked to the orbits of particles

⇒ determine where particles go ⇒ along or against electric fields

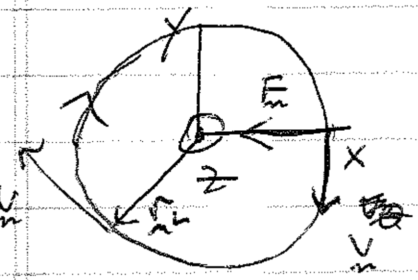
⇒ do particles gain or lose energy?

Uniform magnetic field

$$m \frac{d\mathbf{v}_\perp}{dt} = q\, \mathbf{v}_\perp \times \mathbf{B} \qquad m \frac{dv_\parallel}{dt} = 0$$

$$m \frac{d^2\mathbf{v}_\perp}{dt^2} = \frac{q^2}{m}(\mathbf{v}_\perp \times \mathbf{B}) \times B = -\frac{q^2 B^2}{m} \mathbf{v}_\perp$$

$$\Omega = \frac{qB}{mc} \qquad \frac{d^2}{dt^2}\mathbf{v}_\perp + \Omega^2 \mathbf{v}_\perp = 0$$



$$v_z = \text{const}$$

$$m \frac{v_\theta^2}{r} = \frac{q v_\theta B}{c}$$

r_L = Larmor radius or gyro radius

$$r_L = \frac{v_\theta}{\Omega} \qquad \mathbf{r}_L = -\frac{\mathbf{v} \times \mathbf{\Omega}}{\Omega^2}$$

Uniform Electric and Magnetic Field

$$m \frac{d\underset{\sim}{v}}{dt} = q\left(\underset{\sim}{E} + \frac{\underset{\sim}{v} \times \underset{\sim}{B}}{c}\right) \qquad E_{\parallel} = \frac{\underset{\sim}{E}\cdot\underset{\sim}{B}}{B} = 0$$

define $\quad \underset{\sim}{v} = \tilde{v} + \frac{c\,\underset{\sim}{E}\times\underset{\sim}{B}}{B^2}$

$$m\frac{d\tilde{v}}{dt} = q\left(\underset{\sim}{E} + \frac{\tilde{v}\times B}{c} + \underbrace{\frac{c(E\times B)\times\underset{\sim}{B}}{c\,B^2}}_{-\underset{\sim}{E}}\right)$$

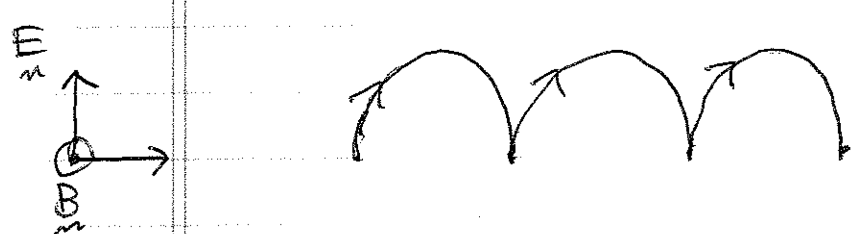
$$= q\,\frac{\tilde{v}\times B}{c}$$

In frame moving with velocity

$$v_E = \frac{c}{B^2}\,\underset{\sim}{E}\times\underset{\sim}{B}$$

there is no electric field.

<u>example</u> particle initially at rest



In moving from particle has $v_L \sim \frac{cE}{B}$

Moves first along $\underset{\sim}{E}$ $\Rightarrow$ gains energy and then deflected by B.

$\Rightarrow$ "pick up" particle in the solar wind

Uniform $\underset{\sim}{B}$ and Varying $\underset{\sim}{E}(t)$

Again take $E_{\parallel} = 0$

$$m \frac{d\underset{\sim}{v}}{dt} = q\left(\underset{\sim}{E} + \frac{1}{c}\underset{\sim}{v}\times\underset{\sim}{B}\right)$$

Again define $\underset{\sim}{v}_{\perp} = \tilde{v}_{\perp} + \frac{c}{B^2}\underset{\sim}{E}\times\underset{\sim}{B}$

$$m\frac{d}{dt}\tilde{\underset{\sim}{v}}_{\perp} + m\frac{c}{B^2}\frac{d\underset{\sim}{E}}{dt}\times\underset{\sim}{B} = \frac{q}{c}\tilde{\underset{\sim}{v}}_{\perp}\times\underset{\sim}{B}$$

If the ~~scale length of $\underset{\sim}{E}$ is long compared with r_L or the~~ time variation of $\underset{\sim}{E}$ is long compared with the cyclotron period can iterate once more

$$\tilde{v}_{\perp} = \tilde{\tilde{v}}_{\perp} + \underset{\sim}{v}_p$$

$$m\frac{d}{dt}\tilde{\tilde{v}}_{\perp} + \xrightarrow{\text{small}}\cancel{m\frac{d}{dt}\underset{\sim}{v}_p} + m\frac{c}{B^2}\frac{d\underset{\sim}{E}}{dt}\times\underset{\sim}{B}$$

$$= \frac{q}{c}\underset{\sim}{v}_p\times\underset{\sim}{B} + \frac{q}{c}\tilde{\tilde{\underset{\sim}{v}}}_{\perp}\times\underset{\sim}{B}$$

$$m\frac{c}{B^2}\,\underbrace{\underset{\sim}{B}\times\left(\frac{d\underset{\sim}{E}}{dt}\times\underset{\sim}{B}\right)}_{\frac{d\underset{\sim}{E}}{dt}B^2} = \frac{q}{c}B^2\underset{\sim}{v}_p$$

$$\underset{\sim}{V}_p = \frac{c}{B}\frac{1}{\Omega}\frac{d}{dt}\underset{\sim}{E}$$

$$m\frac{d}{dt}\tilde{V}_\perp = q\frac{1}{c}\tilde{\underset{\sim}{V}}_\perp \times \underset{\sim}{B} - m\frac{d}{dt}\underset{\sim}{V}_p$$

$\Rightarrow$ polarization drift $\longrightarrow \underset{\sim}{V}_p$

$\Rightarrow$ along direction of $\underset{\sim}{E}$ $\Rightarrow$ energy gain

$\Rightarrow$ smaller than V_E

$$\frac{V_p}{V_E} \sim \frac{1}{\Omega}\frac{d}{dt} \sim \frac{\omega}{\Omega} \ll 1$$

Energy gain of $\underset{\sim}{V}_E$

$$W = \frac{1}{2} m V_E^2$$

$$\dot{W} = \frac{2}{2} m \underset{\sim}{V}_E \cdot \frac{d}{dt}\frac{c\underset{\sim}{E}\times\underset{\sim}{B}}{B^2}$$

$$= m\left(\frac{c}{B^2}\right)^2 (\underset{\sim}{E}\times\underset{\sim}{B})\cdot\left(\frac{d\underset{\sim}{E}}{dt}\times\underset{\sim}{B}\right)$$

$$= \frac{mc^2}{B^2}\underset{\sim}{E}\cdot\frac{d\underset{\sim}{E}}{dt} = \frac{mc^2}{B^2}\underset{\sim}{E}\cdot\frac{B\Omega}{c}\underset{\sim}{V}_p$$

$$= q\underset{\sim}{E}\cdot\underset{\sim}{V}_p$$

$\Rightarrow$ Drift along $\underset{\sim}{E}$ due to $\underset{\sim}{V}_p$ enables $\underset{\sim}{V}_E$ to change (energy to increase)

General particle drifts in ~~a~~ space and time varying fields

Assumptions:

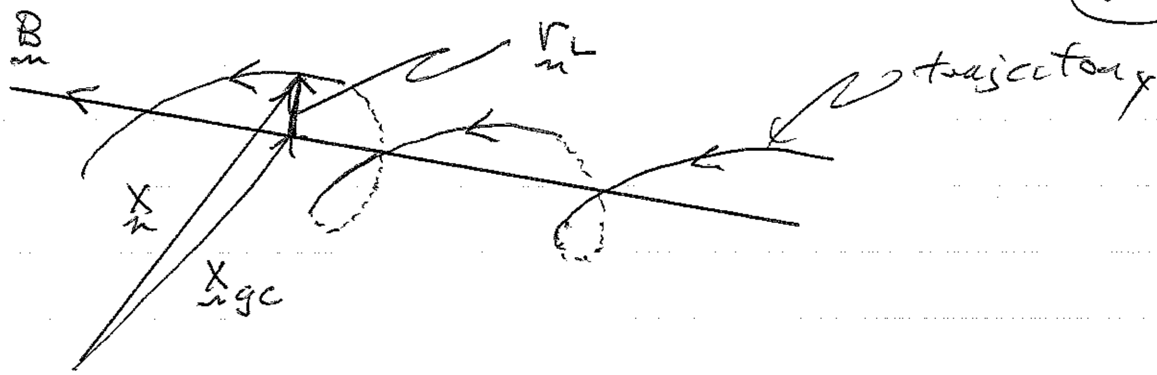
1. slow time variation over gyro time (neglect $\frac{\partial \underset{\sim}{B}}{\partial t}$)
2. weak space variation so little variation over a gyroradius scale
3. $|\underset{\sim}{E}| \ll |\underset{\sim}{B}| \Rightarrow V_E \ll c$

Will have ~~motion~~ gyro motion around $\underset{\sim}{B}$, drifts across $\underset{\sim}{B}$ and motion along $\underset{\sim}{B}$.

Separation of scales allows us to separate the motion into ~~a~~ fast gyro ~~m~~ motion and the slow drifts that are obtained by averaging over the fast motion. We write

$$\underset{\sim}{X}(t) = \underset{\sim}{X}_{gc} + \underset{\sim}{r}_L(t)$$

$$\underset{\sim}{V}(t) = \underset{\sim}{V}_{gc} + \underset{\sim}{V}_L(t)$$



$$\underset{\sim}{X} = \underset{\sim}{X}_{gc} + \underset{\sim}{r}_L$$

Expand the eqn of motion by writing

$$B[X(t)] = B(\underset{\sim}{X}_{gc}) + \underset{\sim}{r}_L \cdot \nabla \underset{\sim}{B}(\underset{\sim}{X}_{gc})$$

$$\equiv \underset{\sim}{B}_{gc} + \underset{\sim}{r}_L \cdot \nabla \underset{\sim}{B}_{gc} + \cdots$$

Same for $\underset{\sim}{E}$

Equation of motion:

$$m(\dot{v}_{gc} + \dot{v}_L) = q\left[\underset{\sim}{E}_{gc} + \underset{\sim}{r}_L \cdot \nabla \underset{\sim}{E}_{gc}\right]$$

$$+ \frac{q}{c}(\underset{\sim}{V}_{gc} + \underset{\sim}{V}_L) \times (\underset{\sim}{B}_{gc} + \underset{\sim}{r}_L \cdot \nabla \underset{\sim}{B}_{gc})$$

Define fast gyro motion by

$$m\dot{V}_L = \frac{q}{c}\underset{\sim}{V}_L \times \underset{\sim}{B}_{gc} + \mathcal{O}(\underset{\sim}{r}_L, \underset{\sim}{V}_L)$$

$\Rightarrow$ subtract and average over gyro orbit.

$$m\dot{V}_{gc} = q\underset{\sim}{E}_{gc} + \frac{q}{c}\underset{\sim}{V}_{gc} \times \underset{\sim}{B}_{gc}$$

$$+ \frac{q}{c}\langle \underset{\sim}{V}_L \times \underset{\sim}{r}_L \cdot \nabla \underset{\sim}{B}_{gc} \rangle$$

~~where terms linear in r_L vanish~~

$\Rightarrow$ evaluate $\underset{\sim}{V}_L$, $\underset{\sim}{r}_L$ to lowest order since quadratic term is already small

Consider

$$\underset{\sim}{r}_L = -\frac{1}{\Omega}\underset{\sim}{V}_L \times \hat{z}$$

$$r_{Lx} = -\frac{V_{Ly}}{\Omega}, \quad r_{Ly} = \frac{1}{\Omega}V_{Lx}$$

$$\langle \underset{\sim}{V}_L \times (\underset{\sim}{r}_L \cdot \nabla)\underset{\sim}{B} \rangle = -\frac{1}{\Omega}\langle V_L \times (V_L \times \hat{z})\cdot\nabla \underset{\sim}{B} \rangle$$

where take $\underset{\sim}{B}$ locally along $\hat{z}$

$$= -\frac{1}{\Omega}\left[\langle V_x^2\rangle\left(-\frac{\partial}{\partial y}\right)\hat{x}\times\underset{\sim}{B} + \langle V_y^2\rangle \frac{\partial}{\partial x}\,\hat{y}\times\underset{\sim}{B}\right]$$

$$= -\frac{V_L^2}{2\Omega}\left(\frac{\partial}{\partial x}\hat{y}\times\underset{\sim}{B} - \frac{\partial}{\partial y}\hat{x}\times\underset{\sim}{B}\right) \qquad \langle V_x^2\rangle = \langle V_y^2\rangle = \frac{1}{2}V_L^2$$

$$= -\frac{V_L^2}{2\Omega}\left[(\hat{z}\times\nabla)\times\underset{\sim}{B}\right] = -\frac{V_L^2}{2\Omega}\left(\nabla B_z\right)$$

$\Rightarrow$

$$m\dot{V}_{gc} = q\underset{\sim}{E}_{gc} + \frac{q}{c}\underset{\sim}{V}_{gc}\times\underset{\sim}{B}_{gc} - \cancel{\frac{q}{c}}\frac{m\cancel{c}V_L^2}{2\cancel{\Omega}\cancel{q}B}\nabla B_{gc}$$

$$\mu \equiv \frac{mV_L^2}{2B}$$

Drop gc subscripts

$$m\dot{\underset{\sim}{V}} = q\underset{\sim}{E} + \frac{q}{c}\underset{\sim}{V}\times\underset{\sim}{B} - \frac{mV_L^2}{2B}\nabla B$$

Separate the $\parallel$ and $\perp$ components

$$\mathbf{v} = v_\parallel \mathbf{b} + \mathbf{v}_\perp \Longleftarrow$$

note: different from $\mathbf{v}_L$

$$\dot{\mathbf{v}} = \frac{d}{dt}\mathbf{v}_\perp + \mathbf{b}\,\frac{dv_\parallel}{dt} + v_\parallel \frac{d}{dt}\mathbf{b}$$

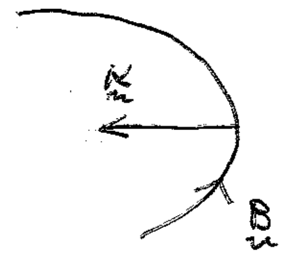
Along $\mathbf{b}$

$$\underbrace{\mathbf{b}\cdot\frac{d}{dt}\mathbf{v}_\perp}_{-\mathbf{v}_\perp\cdot\frac{d}{dt}\mathbf{b}} + \frac{dv_\parallel}{dt} = \frac{q}{m}\,\mathbf{b}\cdot\mathbf{E} - \frac{\mu}{m}\,\mathbf{b}\cdot\nabla B$$

small for $v_E \ll v_t$

$$\boxed{m\frac{dv_\parallel}{dt} = q E_\parallel - \mu\,\mathbf{b}\cdot\nabla B}$$

Across $\mathbf{b}$



$$\frac{d}{dt}\mathbf{b} \simeq v_\parallel\,\mathbf{b}\cdot\nabla\mathbf{b} \equiv v_\parallel\,\boldsymbol{\kappa}$$

again for $v_t \gg v_E$

$\boldsymbol{\kappa}$ = curvature of $\mathbf{B}$

$$\boldsymbol{\kappa}\cdot\mathbf{b} = 0$$

$$m\frac{d}{dt}\mathbf{v}_\perp + v_\parallel^2 m\,\boldsymbol{\kappa} = q\mathbf{E}_\perp + \frac{q}{c}\mathbf{v}_\perp\times\mathbf{B} - \mu\nabla_\perp B$$

Let $\mathbf{F}_\perp = q\mathbf{E}_\perp - \mu\nabla_\perp B - m v_\parallel^2\,\boldsymbol{\kappa}$

$$m \frac{d}{dt} \underset{\sim}{v}_\perp = \underset{\sim}{F}_\perp + \frac{q}{c} \underset{\sim}{v}_\perp \times \underset{\sim}{B}$$

$\Rightarrow$ slow variation

$\Rightarrow$ lowest order

$$\underset{\sim}{F}_\perp + \frac{q}{c} \underset{\sim}{v}_{\perp 0} \times \underset{\sim}{B} = 0$$

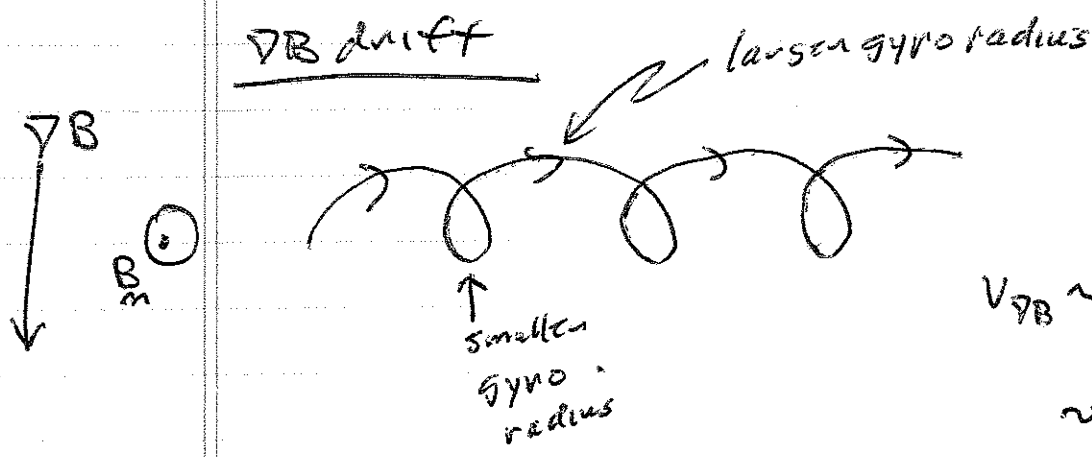
$$\underset{\sim}{v}_{\perp 0} = \cancel{\frac{q}{c}}\, \underset{\sim}{F}_\perp \times \underset{\sim}{B}\, \frac{c}{qB^2}$$

$$\boxed{\underset{\sim}{v}_{\perp 0} = c\,\frac{\underset{\sim}{E} \times \underset{\sim}{B}}{B^2} - \frac{c}{qB}\left(\mu\, \nabla B \times \underset{\sim}{b} + m v_\parallel^2\, \underset{\sim}{\kappa} \times \underset{\sim}{b}\right)}$$

$$V_{\nabla B} = \nabla B \text{ drift} = \frac{c}{qB}\, \mu\, \underset{\sim}{b} \times \nabla B$$

$$V_c = \text{curvature drift} = \cancel{\ } v_\parallel^2\, \underset{\sim}{b} \times \underset{\sim}{\kappa}\, \frac{1}{\Omega}$$

<u>∇B drift</u>

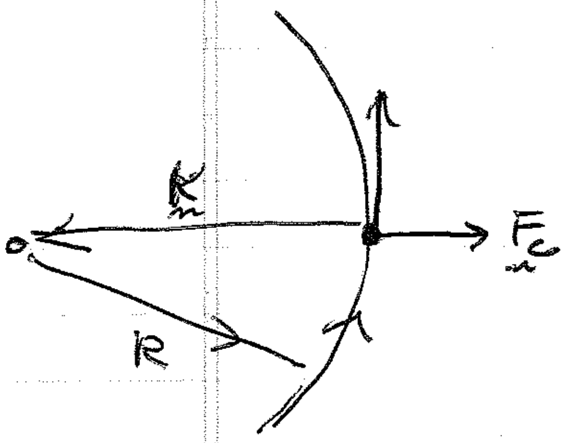


$$V_{\nabla B} \sim \frac{c}{qB}\, \frac{m v_\perp^2}{\cancel{B}}\, \frac{\cancel{B}}{L_B}$$

$$\sim v_\perp\, r_L / L_B \ll v_\perp$$

curvature drift: $\quad V_c \sim \frac{V_\parallel^2}{\Omega}\frac{1}{R_c} \sim v_\perp \frac{\rho_\perp}{R_c}$

$\sim V_{\nabla B}$



$|\underset{\sim}{K}| \sim \frac{1}{R_c}$ $\qquad R_c =$ radius of curvature of field line

$$\underset{\sim}{K} = \underset{\sim}{b}\cdot\nabla\underset{\sim}{b} = \frac{1}{R}\frac{\partial}{\partial\theta}\hat{\theta}$$

$$= -\frac{\hat{r}}{R}$$

$$|K| = \frac{1}{R}$$

In particle reference frame feel an outward centrifugal force

$$\underset{\sim}{F_c} = -m v_\parallel^2 \underset{\sim}{K}$$

$\Rightarrow$ produces $\frac{c}{qB^2}\underset{\sim}{F_c}\times\underset{\sim}{B}$ drift.

Generalized polarization drift at next order.

$$m\frac{d}{dt}\underset{\sim}{v}_{\perp 0} = \frac{q}{c}\underset{\sim}{v}_{\perp 1}\times\underset{\sim}{B}$$

$$\boxed{\underset{\sim}{v}_{\perp 1} = \frac{mc}{qB}\underset{\sim}{b}\times\frac{d}{dt}\underset{\sim}{v}_{\perp 0}}$$

Note that electrons and protons have ∇B and curvature drifts in opposite directions

Conservation of μ

Want to show that μ is an invariant under the assumption of slowly varying fields is ~~assumed~~. made.

$\Rightarrow$ go to the guiding center frame of the particles

$\Rightarrow$

$$m \frac{d}{dt} v_{\parallel} = q E_{\parallel} - \mu \, \underline{b} \cdot \nabla B$$

$\Rightarrow$ construct energy equation

$$\frac{d}{dt} \frac{m v_{\parallel}^2}{2} = q E_{\parallel} v_{\parallel} - v_{\parallel} \mu \, \underline{b} \cdot \nabla B$$

When the magnetic fields are time varying can't transform away $\underline{E}_{\perp}$ since

$$\oint \underline{E} \cdot d\underline{l} \neq 0$$

Need to look at equation of motion Take dot product with $\underline{v}$,

$$\frac{d}{dt}\left(\frac{m v_{\parallel}^2}{2} + \frac{m V_{\perp}^2}{2}\right) = q v_{\parallel} E_{\parallel} + q \underline{v}_{\perp} \cdot \underline{E}_{\perp}$$

$\underline{V}_{L}$ = Larmor velocity

neglect $\quad v_d^2 \ll V_L^2 \sim V_t^2$

$$V_{\perp} \sim V_t \frac{\rho}{L}$$

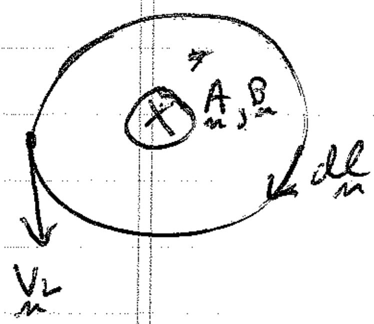
subtract equations

$$\frac{d}{dt}\frac{m v_\perp^2}{2} = q\, \underset{\sim}{v}_\perp \cdot \underset{\sim}{E}_\perp + v_\parallel \mu\, \underset{\sim}{b} \cdot \nabla B$$

Average over Larmor orbit

$$\langle \underset{\sim}{v}_\perp \cdot \underset{\sim}{E}_\perp \rangle = \frac{\Omega}{2\pi} \int dt\, \underset{\sim}{v}_\perp \cdot \underset{\sim}{E}_\perp$$

$$= -\frac{\Omega}{2\pi} \oint d\underset{\sim}{l} \cdot \underset{\sim}{E}_\perp = -\frac{\Omega}{2\pi} \int_A d\underset{\sim}{A} \cdot \nabla \times \underset{\sim}{E}$$



$$\underset{\sim}{v}_\perp dt = -d\underset{\sim}{l}$$

$$\frac{1}{c}\frac{\partial \underset{\sim}{B}}{\partial t} + \nabla \times \underset{\sim}{E} = 0$$

$$\langle \underset{\sim}{v}_\perp \cdot \underset{\sim}{E}_\perp \rangle = +\frac{\Omega}{2\pi}\frac{1}{c} \int d\underset{\sim}{A} \cdot \frac{\partial \underset{\sim}{B}}{\partial t}$$

$$\simeq \frac{\Omega}{2\cancel{\pi} c}\, r_L^2 \cancel{\pi} \frac{\partial}{\partial t} B$$

$$= \frac{v_\perp^2}{2c\Omega}\frac{\partial}{\partial t} B = \frac{v_\perp^2 m \cancel{c}}{2\cancel{c}\, q B}\frac{\partial B}{\partial t}$$

$$\frac{d}{dt}\, m\frac{v_\perp^2}{2} = \frac{m v_\perp^2}{2B}\frac{\partial B}{\partial t} + \frac{m v_\perp^2}{2B} v_\parallel\, b \cdot \nabla B$$

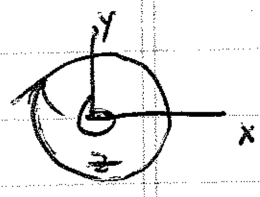
$$= \mu \frac{d}{dt} B$$

$$\frac{d}{dt}\mu B = \mu \frac{d}{dt} B \implies \boxed{\frac{d}{dt}\mu = 0}$$

μ conservation is related to the invariant of

$$J = \oint p\,dq = 2\pi P_\theta$$

Consider ~~a local~~ the local magnetic field taken to be along z.



$$B_z = \hat{z}\cdot\nabla\times\underline{A} = \frac{1}{r}\frac{\partial}{\partial r} r A_\theta$$

$$A_\theta = \frac{B_z}{2} r$$

P_θ = canonical angular momentum

$$= m r v_\theta + \frac{q}{c} A_\theta r \qquad \dot\theta = -\Omega$$

$$= m r^2 \dot\theta + \frac{qB_z}{2c} r^2$$

$$= -\cancel{m} r^2 \frac{qB_z}{\cancel{m}c} + \frac{qB}{2c} r^2 = -r^2 \frac{qB}{2cm} m$$

$$= -m \frac{v_\perp^2}{2\cancel{\Omega}} \frac{mc}{qB}$$

$$= -\frac{mc}{q}\mu$$

$$J = -\frac{2\pi mc}{q}\mu$$

Identity for vacuum fields

$$0 = \underset{\sim}{b} \times (\nabla \times \underset{\sim}{B}) = \nabla B - b \cdot \nabla B \underset{\sim}{b}$$

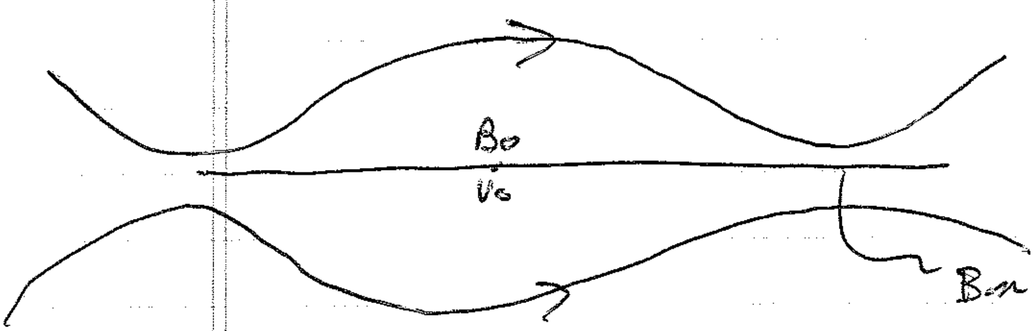
$$= \nabla B - \underset{\sim}{b}\underset{\sim}{b} \cdot \nabla B - B \underset{\sim}{K}$$

$$\Rightarrow \underset{\sim}{K} = \frac{1}{B} \left(\nabla B - \underset{\sim}{b}\underset{\sim}{b} \cdot \nabla \underset{\sim}{B} \right)$$

$$\Rightarrow \underset{\sim}{v}_{\perp 0} = c \frac{\underset{\sim}{E} \times \underset{\sim}{B}}{B^2} + \frac{c}{qB} \left(\mu\, b \times \nabla B + m v_{\parallel}^2 \frac{1}{B}\, b \times \nabla B \right)$$

$$\boxed{\underset{\sim}{v}_{\perp 0} = c \frac{E \times B}{B^2} + \frac{c}{qB^2} \left(B\mu + \frac{m v_{\parallel}^2}{\cancel{B}} \right) \underset{\sim}{b} \times \nabla B}$$

Example: Mirror trapping



Energy cons.

$$\cancel{\tfrac{1}{2}m}\, v_\parallel^2 + \cancel{\tfrac{1}{2}m}\, v_\perp^2 = \cancel{\tfrac{1}{2}m}\, v_0^2$$

$$v_\parallel^2 + \frac{v_\perp^2 m}{2B}\,\frac{B\,2}{m} = v_0^2$$

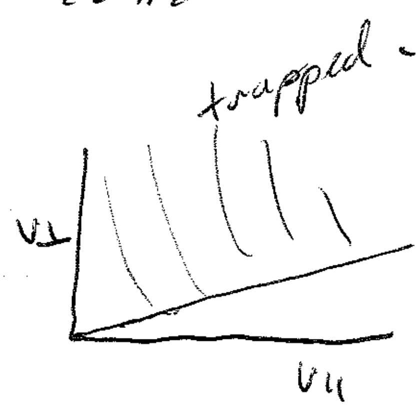
$$v_\parallel^2 + \mu B\,\frac{2}{m} = v_0^2$$

$$v_\parallel^2 = v_0^2 - \mu B\,\frac{2}{m}$$

$$\mu = \text{const} = \frac{v_{\perp 0}^2 m}{2B_0} \longleftarrow \text{values at center}$$

$$v_\parallel^2 = v_0^2 - \frac{B}{B_0}\, v_{\perp 0}^2$$

$$v_\parallel^2 = v_{\parallel 0}^2 + \left(1 - \frac{B}{B_0}\right) v_{\perp 0}^2$$



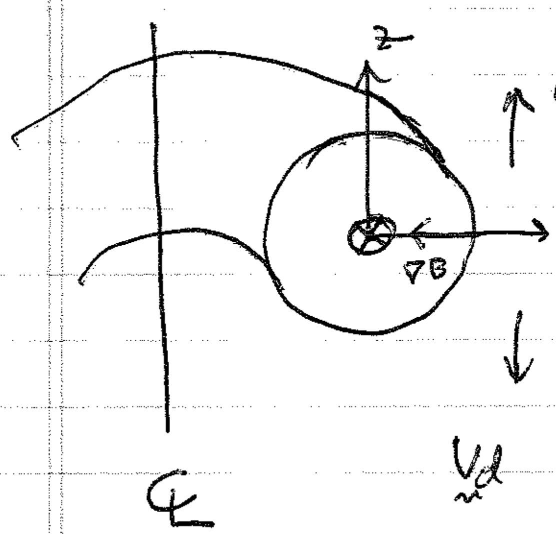
trapping for

$$v_{\parallel 0}^2 + v_{\perp 0}^2\left(1 - \frac{B_m}{B_0}\right) < 0$$

$$\boxed{\frac{B_m}{B_0} > 1 + \frac{v_{\parallel 0}^2}{v_{\perp 0}^2}} \qquad \frac{v_{\parallel 0}}{v_{\perp 0}} < \sqrt{\frac{B_m}{B_0} - 1}$$

Toroidal Orbits

Consider a pure toroidal field



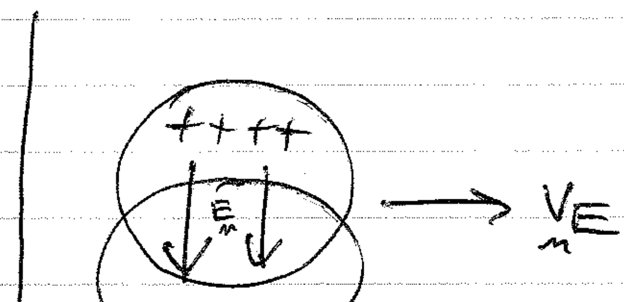
↑ ions

$\underset{\sim}{B} = B_0 R_0 \nabla \varphi$

$B_\varphi \sim \frac{1}{R}$

↓ electrons

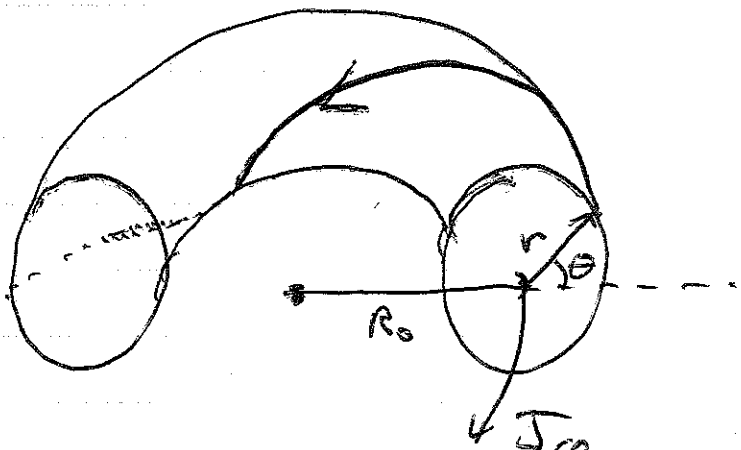
$$\underset{\sim}{V}_d = \frac{c}{q B^2}\left(B\mu + m v_\parallel^2 \right) \underset{\sim}{b} \times \nabla B$$



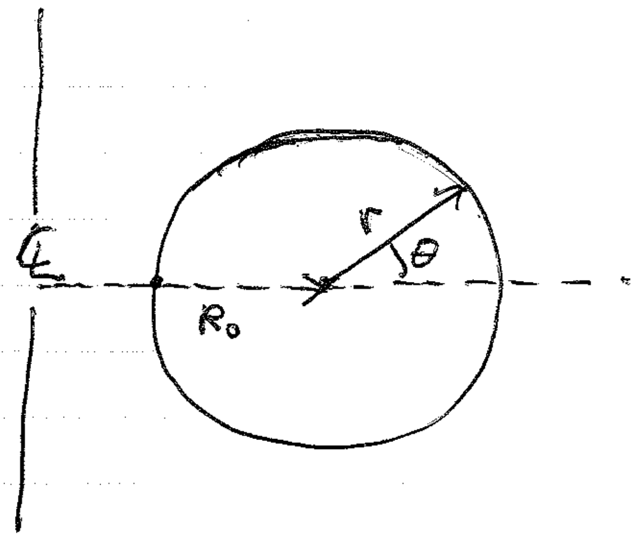
plasma convects radially
outward.

℄

Add poloidal field:



Current comes out of doughnut



Take $\frac{r}{R_0} << 1$

$\Rightarrow$ high aspect ratio

$$\frac{B_\theta}{B_\varphi} \sim \frac{r}{R_0}$$

Follow field line once around torus.

$$\frac{R\,d\varphi}{r\,d\theta} = \frac{B_\varphi}{B_\theta} \qquad B_0 = B_\varphi(R_0)$$

$$\frac{\Delta\varphi}{\Delta\theta} = \frac{r}{R_0}\frac{B_0}{B_\theta} \equiv q = \text{safety factor}$$

$$\Delta\varphi = q\,\Delta\theta$$

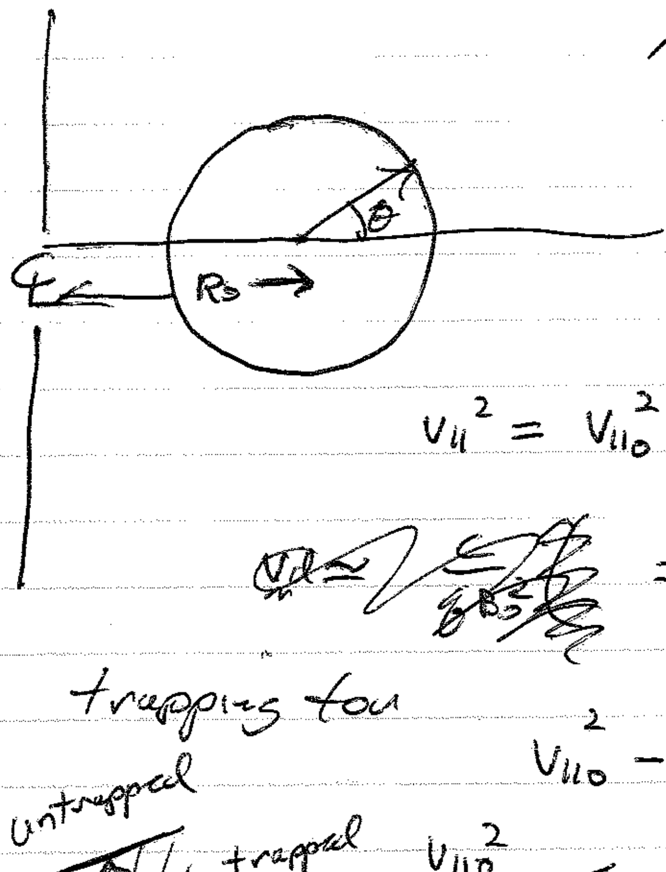
$$\frac{\Delta\varphi}{2\pi} = q\,\frac{\Delta\theta}{2\pi}$$

Let $\Delta\theta = 2\pi$ $\Rightarrow$ once around poloidal direction.

$$\frac{\Delta\varphi}{2\pi} = q$$

q = # of times loop around ~~poloidal d~~ the toroidal direction for a single loop around poloidal direction.

Orbits



$$\cancel{v_\parallel^2 = v_{\parallel 0}^2 \pm \mu(B_0 - B)}$$

$$B = \frac{B_0 R_0}{R_0 + r\cos\theta}$$

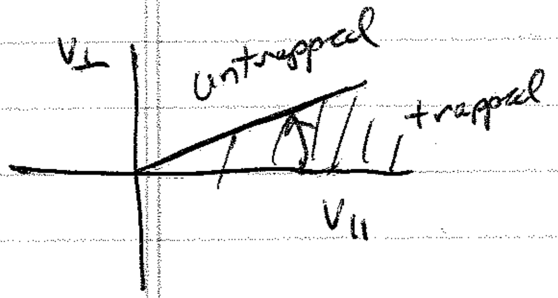
$$\cong B_0\left(1 - \frac{r}{R_0}\cos\theta\right)$$

$$v_\parallel^2 = v_{\parallel 0}^2 + \mu\frac{2}{m}B_0\left(\cancel{1} - \frac{r}{R_0} - \cancel{1} + \frac{r}{R_0}\cos\theta\right)$$

$$= v_{\parallel 0}^2 - v_{\perp 0}^2\frac{r}{R_0}(1 - \cos\theta)$$

trapping for

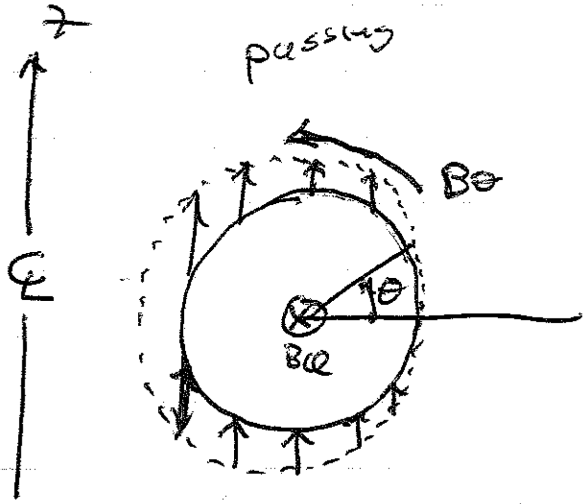
$$v_{\parallel 0}^2 - v_{\perp 0}^2\frac{r}{R_0}\,2 < 0$$

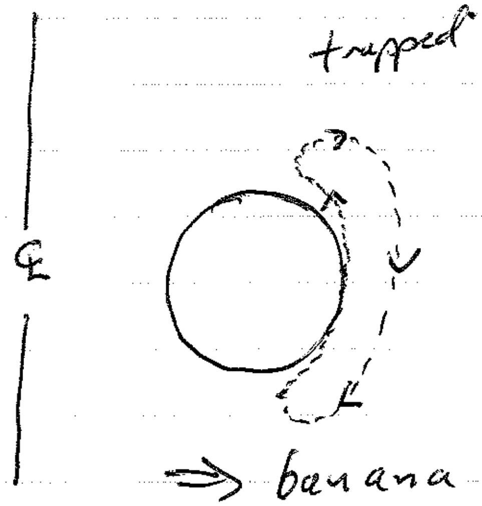


$$\frac{v_{\parallel 0}^2}{v_{\perp 0}^2} < \frac{2r}{R_0} \equiv 2\epsilon$$

$$\frac{v_{\parallel 0}}{v_{\perp 0}} < \sqrt{2\epsilon} \ll 1$$

What about drifts?





$\Rightarrow$ banana orbits

$$\underset{\sim}{b} \times \nabla B \simeq \frac{B_0}{R_0}\hat{z}$$

$$v_d \simeq \frac{c}{q \cancel{B_0}}\left(\mu B_0 + m v_\parallel^2(\theta)\right)\frac{\cancel{B_0}}{R_0}\hat{z}$$

$$v_\parallel(\theta) = v_\parallel(\cos\theta)$$

Find displacement from the circular field line.

$$\Delta r \simeq \int dt\, v_d(\theta)\sin\theta \Rightarrow \begin{array}{l} \text{for } 0<\theta<\pi \quad \dot{\Delta r}>0 \\ \text{for } \pi<\theta<2\pi \quad \dot{\Delta r}<0 \end{array}$$

$$dt = \frac{dt}{r\,d\theta}\, r\,d\theta = \frac{r\,d\theta}{v_\theta} = r\,\frac{B_0}{B_\theta}\,\frac{d\theta}{v_\parallel}$$

$$v_\theta = v_\parallel \frac{B_\theta}{B_0}$$

$$2 v_\parallel \frac{d v_\parallel}{d\theta} = -\frac{v_{\perp 0}^2\, r}{R_0}\sin\theta$$

$$\Delta r = -\int \frac{B_0}{B_\theta}\frac{d\theta}{\cancel{v_\parallel}} V_d(\theta)\, 2\cancel{v_\parallel}\frac{dv_\parallel}{d\theta}\frac{R_0 \cancel{r}}{\cancel{r}\, v_{\perp 0}^2}$$

$$= -\frac{B_0}{B_\theta}\frac{2R_0}{\cancel{r}\, v_{\perp 0}^2}\int d\theta\, V_d(\theta)\frac{dv_\parallel}{d\theta}$$

trapped particles

$$V_d \simeq \frac{c}{q\cancel{B_0}}\frac{\mu \cancel{B_0}}{R_0}\hat{z} \simeq \frac{c}{qR_0}\mu\hat{z}$$

$$\Delta r \simeq -\frac{\cancel{B_0}}{B_\theta}\frac{\cancel{2}\cancel{R_0}}{\cancel{r}\,\cancel{v_{\perp 0}^2}}\frac{c}{q\cancel{R_0}}\frac{\cancel{v_{\perp 0}^2} m}{\cancel{2}\cancel{B_0}}\left[v_\parallel(\theta) - v_\parallel(0)\right]$$

$$\Delta r = -\frac{1}{\Omega_\theta}\left[v_\parallel(\theta) - v_\parallel(0)\right]$$

$$v_\parallel \sim \sqrt{2\epsilon}\, V_{\perp 0}$$

$$\Delta r \sim \frac{\sqrt{2\epsilon}\; B_0 r}{\Omega_0 \; B_\theta R_0}\frac{R_0}{r}v_{\perp 0} \sim q\sqrt{\frac{2}{\epsilon}}\frac{v_{\perp 0}}{\Omega_0}$$

$$\sim \rho_0\, q\sqrt{\frac{2}{\epsilon}} \gg \rho_0$$

$\rho_0 = \dfrac{v_{\perp 0}}{\Omega_0}$ = Larmor radius.

passing particles

$$2\pi r \sim v_\theta t \sim v_\parallel \frac{B_\theta}{B_0} t$$

$$V_d \sim \frac{m v_\perp^2 c}{q R_0 2 B_0} \sim \frac{1}{2}\frac{v_\perp \rho_0}{R_0}$$

$$V_d t \sim \cancel{\tfrac{1}{2}}\frac{\cancel{v_\perp}\rho_0}{R_0}\frac{\cancel{2}\pi r B_0}{B_\theta \cancel{v_\parallel}}$$

$$\sim \pi q \rho_0$$

small