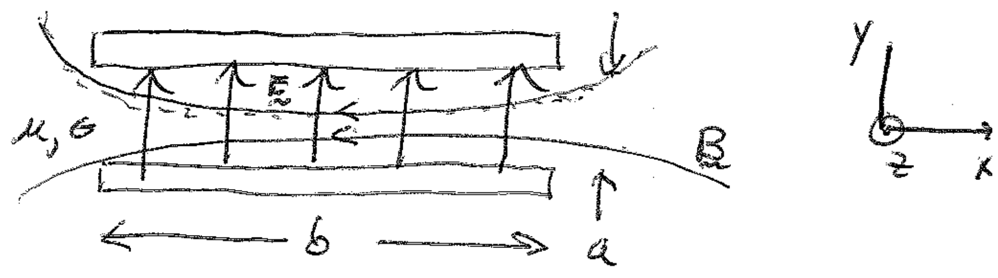
Homework 8 Solutions

~~Hwk # 9 Solutions~~

8.3(a)



$b >> a$

Take $\quad \mathbf{E} = \mathrm{Re}\left(E_0 \hat{j}\, e^{ikz - i\omega t} \right)$

$\quad\quad\quad \mathbf{B} = \mathrm{Re}\left(B_0 \hat{i}\, e^{ikz - i\omega t} \right)$

Both E_0, B_0 remain finite up to the surface of the conductor. We ignore ~~the~~ fringe effects.

$\Rightarrow$ if the conductor were ideal E_0, B_0 would be zero inside the conductor due to a surface charge and a surface current, respectively.

Since $\nabla \cdot \mathbf{E} = 0$ ~~in the~~ between the conductors,

$$\frac{\partial}{\partial y} E_{y0} = 0 \quad \Rightarrow \quad \boxed{E_{y0} = \text{const.}}$$

From ~~Faraday's Law~~ Ampere's Law

$$\nabla \times \mathbf{B} = \mu\epsilon \frac{\partial}{\partial t} \mathbf{E}$$

$$ik B_0 = \mu\epsilon(-i\omega) E_0$$

(1) $$B_0 = -\frac{\mu\epsilon\omega}{k} E_0$$

~~Ampere's Law~~ Faraday's Law

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0$$

$$-i\omega B_{x0} - ik E_{y0} = 0$$

(2) $$B_0 = -\frac{k}{\omega} E_0$$

From 1 and 2

$$-\frac{k}{\omega} E_0 = -\frac{\mu\epsilon\omega}{k} E_0$$

$$\boxed{k^2 = \mu\epsilon\omega^2} \Rightarrow \text{same as homogeneous system}$$

Average power

$$P = \int dA\, \underset{\sim}{S} \cdot \hat{z} = ab \frac{1}{2}\, \hat{z} \cdot \underset{\sim}{E_0} \times \underset{\sim}{B_0^*} \frac{1}{\mu}$$

$$= -ab \frac{1}{2} \frac{1}{\mu} E_{0y} B_{0x}^* = \frac{ab}{2} \frac{1}{\mu} \frac{\omega}{k} |B_0|^2$$

$$= \frac{ab}{2} \frac{\mu}{\sqrt{\mu\epsilon}} |H_0|^2 = \boxed{\frac{ab}{2} \sqrt{\frac{\mu}{\epsilon}} |H_0|^2}$$

Attenuation Rate

3 $$-\frac{\Delta P}{\Delta z} = +2\gamma P$$ $\Delta P =$ Poynting flux ~~change~~ in conductor in distance Δz

inside conductor will have a small E_{zc}.

$$\nabla \times \underset{\sim}{H_c} = \sigma \underset{\sim}{E_c} \Rightarrow \nabla \simeq \frac{\partial}{\partial y} \hat{j}$$

$$-\frac{\partial}{\partial y} H_{0xc} = \sigma E_{0zc}$$

inside conductor $H_{0xc} \sim e^{i\frac{y}{\delta}} e^{-\frac{y}{\delta}}$

two conductors

$$E_{0zc} = \frac{1}{\mu_c \sigma} (-i+1) \frac{1}{\delta} B_{0c}$$

$$\Delta P \doteq 2 \Delta z\, b\, \mathrm{Re}\, \hat{j} \cdot \underset{\sim}{E_0} \times \underset{\sim}{H_0^*} \frac{1}{2}$$

$$= \Delta z\, b\, \mathrm{Re}(E_{0cz} H_{0x}^*) = -\Delta z\, b \frac{1}{\sigma\delta} |H_0|^2 \quad (4)$$

note: H_{0x} is continuous across boundary from 8 and 4 and $P = \frac{\mu}{2} |H_o|^2 ab \frac{\omega}{k}$

$$\frac{b}{\sigma\delta}|H_o|^2 = 2\gamma \frac{|H_o|^2}{2} ab \frac{1}{\sqrt{\mu\epsilon}}\mu$$

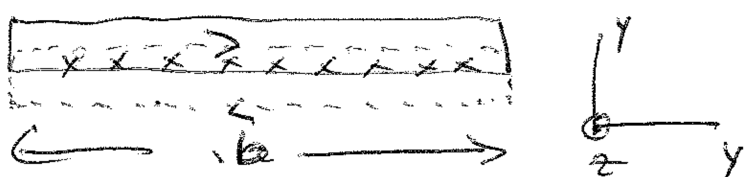
$$\frac{\cancel{b}}{\sigma\delta} = \gamma \mu \frac{a\cancel{b}}{\sqrt{\mu\epsilon}}$$

$$\boxed{\gamma = \sqrt{\frac{\mu}{\epsilon}} \frac{1}{\sigma\delta a}}$$

Impedance

The impedance of any guide is the ratio of the voltage across the guide at any location to the current in one of conductors

$$Z_0 = \frac{E_{yo}\, a}{I_o} \qquad \nabla \times H = J$$



integrating area of loop

$$H_{oy}\, b = I_o \qquad Z_0 = \frac{E_{yo}\, a}{H_{oy}\, b}$$

$$Z_0 = \frac{\omega}{k}\frac{B_o\, a}{H_o\, b} = \frac{1}{\sqrt{\mu\epsilon}}\mu \frac{a}{b} = \boxed{\sqrt{\frac{\mu}{\epsilon}}\frac{a}{b}}$$

Inductance per unit length

Define inductance in terms of stored energy

$$\frac{1}{2} L \frac{|I_o|^2}{2} = \underbrace{\frac{|B_o|^2}{2\mu} ab \frac{1}{2}}_{}$$

mag. $\frac{\text{energy}}{\text{unit length}}$.

$$\cancel{\frac{1}{2}} L \frac{\cancel{|H_o|^2} b^2}{\cancel{2}} = \frac{\mu^2 \cancel{|H_o|^2}}{\cancel{2\mu}} \frac{a\cancel{b}}{\cancel{2}}$$

$$\boxed{L = \mu \frac{a}{b}}$$

Resistance per unit length

Calculate based on dissipation per unit length

$$\frac{1}{2} |I_o|^2 R = \frac{b}{\sigma\delta} |H_o|^2 (\cancel{2})$$

two conductors

$$\frac{1}{2} b^2 \cancel{|H_o|^2} R = \frac{\cancel{b}}{\sigma\delta} \cancel{|H_o|^2} \cancel{2}$$

$$\boxed{R = \frac{2}{b\sigma\delta}}$$

2 Consider a cylindrical waveguide and TM mode $\Rightarrow E_z \neq 0$

$$\nabla \cdot \underset{\sim}{E} = 0 \Rightarrow \frac{\partial}{\partial z} E_z + \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho E_\rho = 0$$

$$\Rightarrow E_\rho \neq 0$$

$$\frac{\partial \underset{\sim}{B}}{\partial t} + \nabla \times \underset{\sim}{E} = 0$$

$\Rightarrow \hat{\varphi}$ component

$$\frac{\partial}{\partial t} B_\varphi + \frac{\partial}{\partial z} E_\rho - \frac{\partial}{\partial \rho} E_z = 0$$

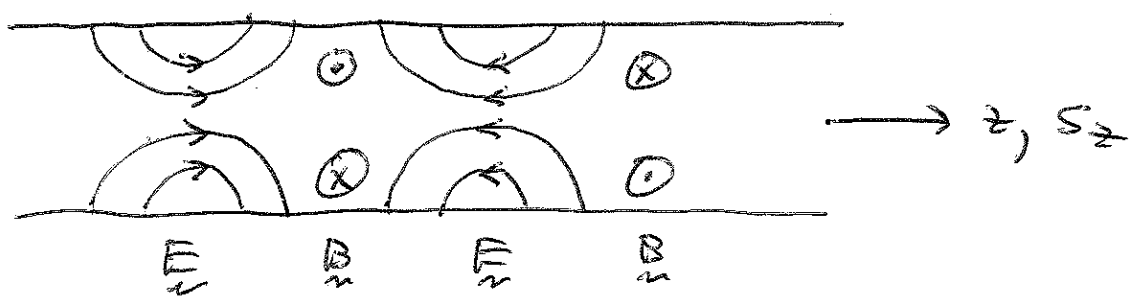
$$\Rightarrow B_\varphi \neq 0$$

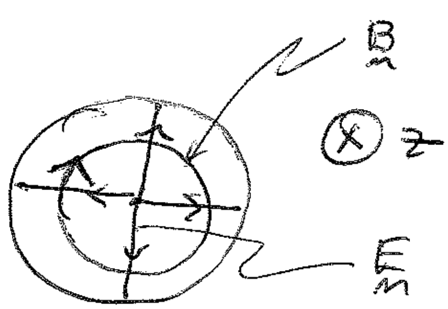
<u>BCs</u>

$E_z = 0$ at $\rho = a$

$E_\rho \neq 0$

$B_\varphi \neq 0$





b) $$\nabla \times \underset{\sim}{B} = \mu\epsilon \frac{\partial}{\partial t}\underset{\sim}{E}$$

$$\frac{\partial \underset{\sim}{B}}{\partial t} + \nabla \times \underset{\sim}{E} = 0$$

$$\nabla \times \frac{\partial \underset{\sim}{B}}{\partial t} = \underbrace{-\nabla \times (\nabla \times \underset{\sim}{E})}_{\nabla^2 \underset{\sim}{E}} = \mu\epsilon \ddot{\underset{\sim}{E}}$$

$$E_z = E_{z0}\, e^{ikz} e^{-i\omega t}$$

$$\nabla^2 E_z = -\mu\epsilon\omega^2 E_z$$

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho} E_{z0} - k_{\cancel{z}}^2 E_{z0} + \mu\epsilon\omega^2 E_{z0} = 0$$

$$\rho\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho} E_{z0} + (\mu\epsilon\omega^2 - k_{\cancel{z}}^2)\rho^2 E_{z0} = 0$$

$\Rightarrow$ Bessel Eqn.

$$\gamma^2 \equiv \mu\epsilon\omega^2 - k^2$$

c)

$$E_{z0} = E_0 J_0(\gamma\rho)$$

$$\boxed{E_{z0} = 0 \text{ at } \rho = a}$$

$$\gamma a = x_{01}$$

$$\gamma = \frac{x_{01}}{a}$$

$$\boxed{k^2 = \mu\epsilon\omega^2 - \frac{x_{01}^2}{a^2}}$$

group velocity

$$\cancel{2}k = \mu\epsilon\,\cancel{2}\omega v_g$$

$$v_g = \frac{k}{\omega}\frac{1}{\mu\epsilon} = \frac{\sqrt{\mu\epsilon\omega^2 - \frac{x_{01}^2}{a^2}}}{\omega\,\mu\,\epsilon}$$

<u>ω large</u>

$$v_g \cong \frac{1}{\sqrt{\mu\epsilon}} = \frac{c}{n} = \text{free space velocity}$$

<u>ω small</u>

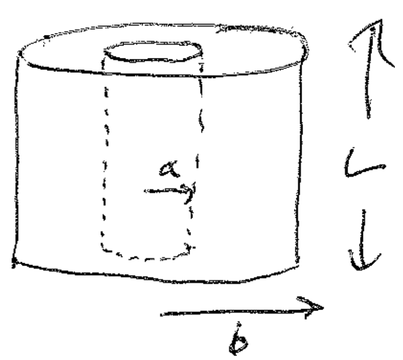
$$v_g = \frac{i\,\frac{x_{01}}{a}}{\omega\mu\epsilon} \Rightarrow \text{no propagation}$$

<u>lowest propagation frequency</u>

$$\omega = \frac{x_{01}}{a}\frac{1}{\sqrt{\mu\epsilon}} = \frac{x_{01}}{a}\frac{c}{n}$$

$$\boxed{\omega = \frac{x_{01}}{a}\frac{c}{n}}$$

3 <u>Resonance cavity</u> $\Rightarrow$ take $\frac{\partial}{\partial\phi} = 0$



Lowest order TE mode

$\Rightarrow B_z \neq 0$

$$\nabla\cdot\underset{\sim}{B} = 0 = \frac{\partial}{\partial z}B_z + \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho B_\rho$$

$\Rightarrow B_\rho \neq 0$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}B_z + \frac{\partial^2}{\partial z^2}B_z + \mu\epsilon\omega^2 B_z = 0$$

$$\nabla \times \mathbf{B} = \mu\epsilon \frac{\partial}{\partial t} \mathbf{E}$$

$\Rightarrow$ φ component

$$\frac{\partial}{\partial z} B_\rho - \frac{\partial}{\partial \rho} B_z = \mu\epsilon \frac{\partial}{\partial t} E_\varphi$$

<u>BCs</u>

$B_\rho = 0$ at $\rho = a$

$E_\varphi = 0$ at $\rho = a$

$$\Rightarrow \boxed{\frac{\partial}{\partial \rho} B_z = 0 \text{ at } \rho = a}$$

$$\boxed{B_z = 0 \text{ at } z = 0, L}$$

$$\Rightarrow B_z \sim \sin\left(\frac{\pi z}{L}\right) e^{-i\omega t}$$

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho} B_z + \underbrace{\left(\mu\epsilon\omega^2 - \frac{\pi^2}{L^2}\right)}_{\gamma^2} B_z = 0$$

$\Rightarrow$ Bessel's Equ.

$$B_z \sim J_0(\gamma\rho),\ N_0(\gamma\rho)$$

$\Rightarrow$ can't discard N_0 since is bounded away from $\rho = 0$

$\Rightarrow$ choose combination of J_0, N_0 to satisfy BC at $\rho = a$

$$B_z = B_0\left[\frac{J_0(\gamma\rho)}{J_0'(\gamma a)} - \frac{N_0(\gamma\rho)}{N_0'(\gamma a)}\right] \sin\left(\frac{\pi z}{L}\right) e^{-i\omega t}$$

$\Rightarrow \frac{\partial B_z}{\partial \rho} = 0$ at $\rho = a$

At $\rho = b$ must have

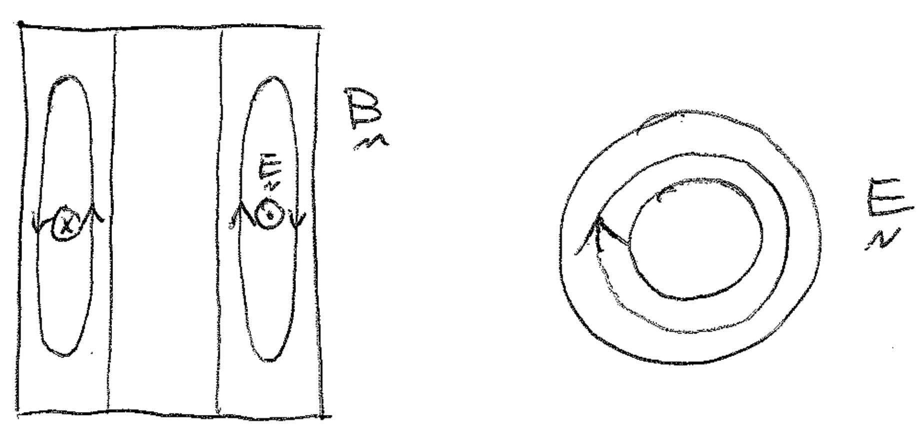
$$\boxed{\frac{J_0'(\gamma b)}{J_0'(\gamma a)} = \frac{N_0'(\gamma b)}{N_0'(\gamma a)}}$$

$\Rightarrow$ defines γ

$$\omega^2 = \left(\frac{\pi^2}{L^2} + \gamma^2\right)\frac{1}{\mu\epsilon}$$

$$\boxed{\omega = \sqrt{\frac{\pi^2}{L^2}+\gamma^2}\;\frac{c}{n}}$$

b)



$$B_z = B_0\, G_0(\rho\gamma)\sin\left(\frac{\pi z}{L}\right)e^{-i\omega t}$$

$$G_0(\rho\gamma) = \frac{J_0(\gamma\rho)}{J_0'(\gamma a)} - \frac{N_0(\gamma\rho)}{N_0'(\gamma a)}$$

$$\Rightarrow \quad B_\rho \sim \cos\left(\frac{\pi z}{L}\right), \quad E_\varphi \sim \sin\left(\frac{\pi z}{L}\right)$$

c) Estimate the Q of the cavity

$$Q \sim \frac{\text{Volume}}{\text{area} \times \delta} \sim \frac{\cancel{\pi}(b^2-a^2)L}{\left[2\cancel{\pi}(b+a)L + \cancel{\pi}(b^2-a^2)\right]\delta}$$

$$\boxed{Q \sim \frac{b-a}{\left(2 + \frac{b-a}{L}\right)\delta}}$$