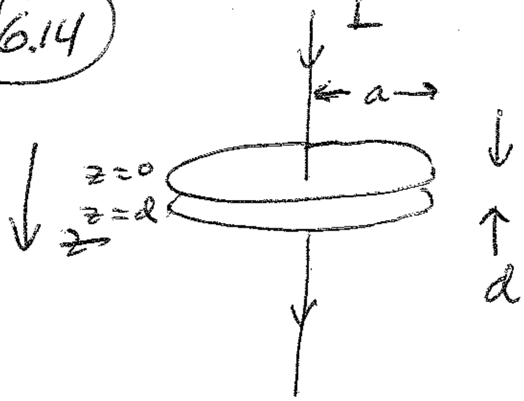
Homework 7 Solutions

6.14



$I = I_0 \cos(\omega t)$

$d \ll a \qquad Q_0 = \frac{\sin\omega t \, I_0}{\omega}$

expansion parameter

$\delta \equiv \frac{\omega a}{c}$

$\Rightarrow$ light propagation time is short compared with frequency

a) <u>Maxwell Eqns</u> $\quad \epsilon = \epsilon_0, \ \mu = \mu_0$

$$\nabla \times \underset{\sim}{B} = \mu_0 \underset{\sim}{J} + \mu_0 \epsilon_0 \frac{\partial}{\partial t} \underset{\sim}{E}$$

$$\nabla \times \underset{\sim}{E} + \frac{\partial}{\partial t} \underset{\sim}{B} = 0$$

$$\nabla \cdot \underset{\sim}{E} = \rho/\epsilon_0$$

$$\nabla \cdot \underset{\sim}{B} = 0$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \underset{\sim}{J} = 0 \quad \Rightarrow \text{ conductor inner surface}$$

$\Rightarrow$ treat $\frac{\omega a}{c}$ as the small parameter and solve the equations order by order in this small parameter.

<u>Lowest order</u> (zero order)

$$\nabla \times \underset{\sim}{B}_0 = \mu_0 \underset{\sim}{J}_0$$

$$\nabla \times \underset{\sim}{E}_0 = 0$$

$$\nabla \cdot \underset{\sim}{E}_0 = \rho_0/\epsilon_0$$

$$\nabla \cdot \underset{\sim}{B}_0 = 0 \qquad \nabla \cdot \underset{\sim}{J}_0 = 0$$

$\underset{\sim}{J}_o$ is current on ~~cond~~ inner surface of the capacitor

$$\Rightarrow \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho J_{o\rho} = 0$$

$$\rho J_{o\rho} = \text{const} \Rightarrow \boxed{J_{o\rho} = 0}$$

$$\Rightarrow \nabla \times \underset{\sim}{B}_o = 0 \text{ and } \nabla \cdot \underset{\sim}{B}_o = 0$$

$$\Rightarrow \boxed{\underset{\sim}{B}_o = 0}$$

$$\nabla \times \underset{\sim}{E}_o = 0 \Rightarrow \underset{\sim}{E}_o = -\nabla\varphi_o$$

$$\hat{n} \times \underset{\sim}{E}_o = 0 \Rightarrow \frac{\partial\varphi_o}{\partial\rho} = 0$$

$\Rightarrow$ ideal conductor

$$\Rightarrow \frac{\partial^2}{\partial z^2}\varphi_o = -\frac{\rho_o}{\epsilon_o} \Rightarrow \rho_o = \rho_o(z) = \sigma_o\,\delta(z)$$

$$\sigma_o = \frac{Q_o}{\pi a^2} = \frac{I_o}{\pi a^2}\frac{\sin\omega t}{\omega}$$

$$\boxed{E_{zo} = \frac{\sigma_o}{\epsilon_o} = \frac{I_o}{\epsilon_o \pi a^2}\frac{\sin\omega t}{\omega}}$$

First order

$$\nabla \times \underset{\sim}{B}_1 = \cancel{\mu_o \underset{\sim}{J}_1} + \mu_o\epsilon_o\frac{\partial}{\partial t}\underset{\sim}{E}_o$$

$$\left.\begin{aligned} &\nabla \times \underset{\sim}{E}_1 + \cancel{\frac{\partial}{\partial t}\underset{\sim}{B}_o}^{\,0} = 0 \\ &\nabla \cdot \underset{\sim}{E}_1 = \rho_1/\epsilon_o = 0 \end{aligned}\right\} \quad \underset{\sim}{E}_1 = 0$$

$$\nabla \cdot \underset{\sim}{B}_1 = 0$$

$$\frac{\partial \varrho_0}{\partial t} + \nabla \cdot \underset{\sim}{J}_1 = 0$$

<u>in vacuum</u> $$\left(\nabla \times \underset{\sim}{B}_1\right)_z = \mu_0 \epsilon_0 \frac{\partial}{\partial t} E_{0z}$$

$$\frac{1}{\varrho}\frac{\partial}{\partial \varrho}\varrho B_{1\varphi} = \mu_0 \cancel{\epsilon_0} \frac{\cancel{\omega} I_0}{\pi a^2} \frac{\cos\omega t}{\cancel{\omega \epsilon_0}}$$

$$\boxed{B_{1\varphi} = \mu_0 \frac{I_0}{\pi a^2} \frac{\varrho}{2} \cos\omega t}$$

note: $B_{1\varphi} = 0$ inside cap. plate

$\Rightarrow$ surface current $J_{1\varrho} \neq 0$

At the surface

$$\left(\nabla \times \underset{\sim}{B}_{1\varphi}\right)_\varrho = \mu_0 J_{1\varrho}$$

$$\underbrace{-\frac{\partial}{\partial z} B_{1\varphi}}_{-B_{1\varphi}\,\delta(z)} = \mu_0 J_{1\varrho}$$

$$J_{1\varrho} = -\frac{1}{\mu_0} B_{1\varphi}\,\delta(z)$$

check continuity of charge

$$\frac{\partial \varrho_0}{\partial t} + \frac{1}{\varrho}\frac{\partial}{\partial \varrho}\varrho J_{1\varrho} = 0$$

$$\frac{I_0}{\pi a^2}\cancel{\delta(z)}\cos\omega t + \frac{1}{\varrho}\frac{\partial}{\partial \varrho}\left(-\frac{\varrho}{\mu_0} B_{1\varphi}\,\cancel{\delta(z)}\right) = 0$$

$$\cancel{\frac{I_0}{\pi a^2}\cos\omega t} - \cancel{\frac{\mu_0 I_0}{\pi a^2}\cos\omega t \frac{1}{\mu_0}} = 0 \qquad \text{OK}$$

<u>second order</u>

$$\nabla \times \underset{\sim}{B}_2 = \mu_0 J_2 + \cancel{\mu_0 \epsilon_0 \frac{\partial}{\partial t} \underset{\sim}{E}_1}^{\,0}$$

$$\nabla \times \underset{\sim}{E}_2 + \frac{\partial}{\partial t} \underset{\sim}{B}_1 = 0$$

$$\nabla \cdot \underset{\sim}{E}_2 = \rho_2 / \epsilon_0$$

$$\nabla \cdot \underset{\sim}{B}_2 = 0$$

$$\cancel{\frac{\partial \rho_1}{\partial t}}^{\,0} + \nabla \cdot \underset{\sim}{J}_2 = 0 \quad \Rightarrow \underset{\sim}{J}_2 = 0$$

$$\Rightarrow \left.\begin{array}{l} \nabla \times \underset{\sim}{B}_2 = 0 \\ \nabla \cdot \underset{\sim}{B}_2 = 0 \end{array}\right\} \underset{\sim}{B}_2 = 0$$

since $\hat{z} \times \underset{\sim}{E}_2 = 0$ at $z = 0, d$

$$\Rightarrow E_{2\rho}, E_{2\varphi} = 0$$

$$\Rightarrow E_{2z} \neq 0$$

$$-\frac{\partial}{\partial \rho} E_{2z} + \frac{\partial}{\partial t} B_{1\varphi} = 0$$

$$-\frac{\partial}{\partial \rho} E_{2z} + \left(-\frac{\mu_0 I_0}{\pi a^2} \frac{\rho}{2} \omega \sin \omega t \right) = 0$$

$$E_{2z} = -\frac{\mu_0 I_0}{\pi a^2} \omega \frac{\rho^2}{4} \sin \omega t + E_{2z}(0)$$

$$E_{2z} = -E_{z0} \frac{\omega^2}{c^2} \rho^2 \frac{1}{4} + E_{2z}(0)$$

$$= \frac{\sigma_2}{\epsilon_0}$$

but $\int da\, \sigma_2 = 0 = E_{2z}(0) \pi a^2 - E_{z0} \frac{\omega^2}{c^2} \frac{2\pi a^4}{16}$

$$E_{2z}(0) = E_{z0} \frac{\omega^2}{c^2} a^2 \frac{1}{8}$$

$$\boxed{E_{2z}(\rho) = E_{z0} \frac{\omega^2}{c^2} \frac{1}{8}\left(a^2 - 2\rho^2\right)}$$

b) time averaged stored energy

$$W_E = \frac{1}{2}\epsilon_0 \int d\underset{\sim}{x} \langle E^2 \rangle$$

$$= \frac{1}{2}\epsilon_0 \frac{I_0^2}{\epsilon_0^2 \pi^2 a^4} \frac{1}{\omega^2} \frac{1}{2} \pi a^2 d$$

$$= \frac{I_0^2 d}{4\pi\epsilon_0 \omega^2 a^2}$$

$$W_B = \frac{1}{2\mu_0} \int d\underset{\sim}{x} \langle B^2 \rangle$$

$$= \frac{1}{\cancel{2\mu_0}} \frac{\mu_0^{\cancel{2}} I_0^2}{\pi^{\cancel{2}} a^4} \frac{1}{4} \frac{1}{\cancel{2}} d\, \cancel{2\pi} \underbrace{\int_0^a d\rho\,\rho\,\rho^2}_{a^4/4}$$

$$\boxed{W_B = \frac{\mu_0 I_0^2 d}{32\pi}}$$

c) capacitance $\frac{1}{2} C \langle V^2 \rangle = W_E$

$$\frac{1}{2} C \frac{d^2 \cancel{I_0^2}}{\epsilon_0^{\cancel{2}} \pi^{\cancel{2}} a^{\cancel{4}2}} \frac{1}{2} \frac{1}{\cancel{\omega^2}} = \frac{\cancel{I_0^2} d}{4\cancel{\pi}\cancel{\epsilon_0}\cancel{\omega^2}\cancel{a^2}}$$

$$\boxed{C = \frac{\epsilon_0 \pi a^2}{d}}$$

$$\frac{1}{2} L\langle I^2\rangle = W_B$$

$$\frac{1}{2} L \frac{\cancel{I_0^2}}{2} = \frac{\mu_0 \cancel{I_0^2} d}{32\pi}$$

$$\boxed{L = \frac{\mu_0 d}{8\pi}}$$

<u>resonance frequency</u>

$$\frac{1}{2} L I^2 + \frac{1}{2} C V^2 = const$$

$$L I \dot{I} + \cancel{2}\, C V \dot{V} = 0$$

$$V = \underline{Ed} = \frac{Q}{A\epsilon_0} d$$

$$L \cancel{I} \ddot{Q} + C \frac{Q}{A\epsilon_0} d \frac{\cancel{I} d}{A\epsilon_0} = 0$$

$$\ddot{Q} + \frac{C d^2}{A^2 \epsilon_0^2 L} Q = 0$$

$$\omega_{res} = \sqrt{\frac{C}{L}} \frac{d}{A\epsilon_0}$$

$$= \sqrt{\frac{\cancel{A}\epsilon_0 \cancel{d^2}\, 8\pi}{\cancel{d}\,\mu_0 \cancel{d}}}\ \frac{\cancel{d}}{\cancel{A}a^2\epsilon_0} = \boxed{\frac{c}{a} 2\sqrt{2}}$$

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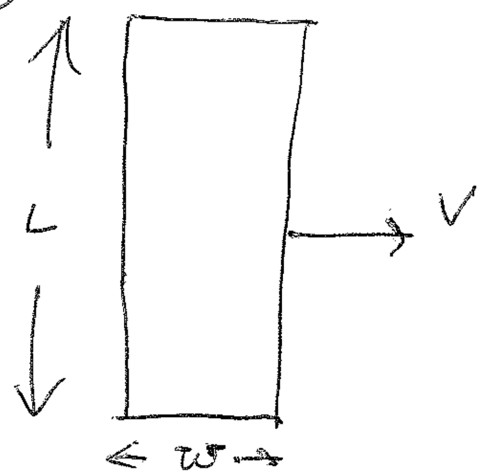
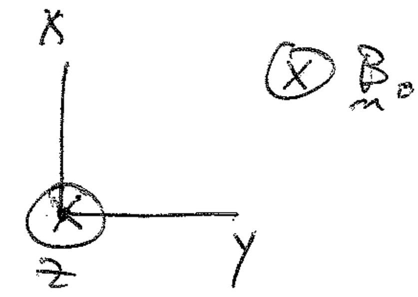




λ = mass / length

a) Since the wire is a perfect conductor, the emf must remain zero so that I in the loop is finite. The magnetic flux cutting through the loop by B_o must be balanced by that due to the current I.

The magnetic field from a wire is given by

$$B = \frac{\mu_0 I}{2\pi r}$$

Flux from one side

$$\Psi = L \int_a^w dr \, \frac{\mu_0 I}{2\pi r} = L \frac{\mu_0 I}{2\pi} \ln\left(\frac{w}{a}\right)$$

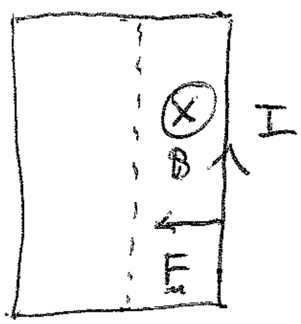
total flux is twice this. Note current is on wire surface so no flux inside wire matching fluxes

$$B_0 \cancel{L} y = \frac{\cancel{2}\cancel{L}\mu_0 I}{\cancel{2}\pi}\ln\left(\frac{w}{a}\right)$$

where y is the distance the loop has penetrated the flux. The current is counter clockwise

$$\boxed{I = \frac{\pi B_0 y}{\mu_0 \ln\left(\frac{w}{a}\right)}}$$

b) $\underset{\sim}{F} = I \underset{\sim}{L} \times \underset{\sim}{B}$ with $\underset{\sim}{B}$ only the external field B_0.



$\underset{\sim}{F}$ is to the left

$$F = I L B_0$$

$$\frac{dF}{dL} = \frac{\pi B_0^2 \cancel{L}}{\mu_0 \ln\left(\frac{w}{a}\right)} y$$

$$\boxed{\lambda \ddot{y} = -\frac{\pi B_0^2}{\mu_0 \ln\left(\frac{w}{a}\right)} y}$$

c) Let $\Omega^2 = \frac{\pi B_0^2}{\lambda \mu_0 \ln \frac{w}{a}}$

Let the loop enter B_0 at $t=0$

$$y = y_0 \sin \Omega t$$

$$\dot{y} = v_y = \Omega y_0 \cos \Omega t$$

& $v_y = V_0$ at $t=0$

$$\Rightarrow y_0 = \frac{V_0}{\Omega}$$

$$\boxed{\begin{aligned} y &= \frac{V_0}{\Omega} \sin \Omega t \\ v_y &= V_0 \cos \Omega t \end{aligned}}$$

solution valid until the loop completely enters the field or until it returns to $y=0$

<u>low velocity</u> $\quad \frac{V_0}{\Omega} < w$

At $\Omega t = \pi$ the loop will return to $y=0$ and exit the field with a velocity $-V_0$

high velocity $\quad \frac{V_0}{\Omega} > \omega$

At

$$\frac{V_0}{\Omega} \sin(\Omega \tau) = \omega$$

the loop will completely enter the loop and the loop will ~~move~~ remain at a constant velocity given by

$$V_y = V_0 \left(1 - \sin^2 \Omega \tau\right)^{\frac{1}{2}}$$

$$= V_0 \left(1 - \frac{\omega^2}{V_0^2} \Omega^2\right)^{\frac{1}{2}}$$

$$\boxed{V_{y\,final} = \left(V_0^2 - \omega^2 \Omega^2\right)^{\frac{1}{2}}}$$