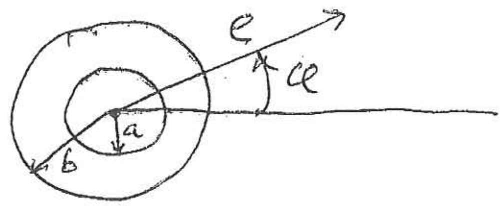
Hwk # 6 Solutions

(1) Jackson 5.14

Hollow cylinder in an initially uniform magnetic field B_0. Solve for $\underset{\sim}{B}$ everywhere.



$\Rightarrow$ no free currents

$\Rightarrow \nabla \times \underset{\sim}{H} = 0$

$$\underset{\sim}{B} = \mu \underset{\sim}{H} \qquad \underset{\sim}{H} = -\nabla \Phi_m$$

$\Rightarrow H_\varphi$ continuous at a, b

$$\nabla \cdot \underset{\sim}{B} = 0 = -\nabla \cdot \mu \nabla \Phi_m$$

$\Rightarrow \Phi_m$ continuous

$$\Rightarrow \nabla^2 \Phi_m \text{ except at } \rho = a, b$$

$\rho > b$

$$\Phi_m = \sum_l c_l \left(\frac{b}{\rho}\right)^l \cos(l\varphi) - \hat{B}_0 \rho \cos\varphi$$

$$\hat{B}_0 \equiv B_0/\mu_0$$

$\Rightarrow$ even in φ

$\Rightarrow$ bounded at ∞ except for B_0

$\Rightarrow$ as in similar electrostatic problems only $l = 1$ survives the matching

$$\Phi_m = c_1 \left(\frac{b}{\rho}\right) \cos\varphi - \hat{B}_0 \rho \cos\varphi$$

$a < \rho < b$

$$\Phi_m = d_1 \left(\frac{b}{\rho}\right) \cos\varphi + e_1 \left(\frac{\rho}{b}\right) \cos\varphi$$

$\rho < a$

$$\phi_m = f_1\left(\frac{\rho}{a}\right)\cos\phi$$

matching

continuity of ϕ_m at b:

1 $$c_1 - B_0 b = d_1 + e_1$$

continuity of ϕ_m at a:

2 $$f_1 = d_1\left(\frac{b}{a}\right) + e_1\left(\frac{a}{b}\right)$$

continuity of B_ρ at b:

3 $$c_1\left(-\frac{1}{b}\right) - B_0 = \left(-d_1\frac{1}{b} + e_1\frac{1}{b}\right)\frac{\mu}{\mu_0}$$

continuity of B_ρ at a;

4 $$f_1\frac{1}{a} = \left(-d_1\frac{b}{a^2} + e_1\frac{1}{b}\right)\frac{\mu}{\mu_0}$$

combining 2 and 4

$$\cancel{d_1\left(\frac{b}{2a}+\frac{b}{a}\right)+e_1\left(\frac{a}{b}\right)}$$

$$d_1\frac{b}{a}\left(1+\frac{\mu}{\mu_0}\right) + e_1\frac{a}{b}\left(1-\frac{\mu}{\mu_0}\right) = 0$$

$$d_1 = e_1\frac{a^2}{b^2}\,\frac{\mu-\mu_0}{\mu+\mu_0}$$

from 1 and 3

$$-2b\hat{B}_0 = d_1\left(1-\frac{\mu}{\mu_0}\right) + e_1\left(1+\frac{\mu}{\mu_0}\right)$$

$$= e_1 \frac{\mu+\mu_0}{\mu_0}\left[1-\frac{a^2}{b^2}\frac{(\mu-\mu_0)^2}{(\mu+\mu_0)^2}\right]$$

Let $G \equiv \frac{\mu+\mu_0}{2\mu_0}\left[1-\frac{a^2}{b^2}\frac{(\mu-\mu_0)^2}{(\mu+\mu_0)^2}\right]$

$$e_1 = -\frac{b\hat{B}_0}{G} \qquad d_1 = -\frac{b\hat{B}_0}{G}\frac{a^2}{b^2}\frac{\mu-\mu_0}{\mu+\mu_0}$$

$\Rightarrow$ some algebra $\quad c_1 = \frac{\hat{B}_0 b}{G}\left(1-\frac{a^2}{b^2}\right)\frac{\mu-\mu_0}{2\mu_0}$

$$f_1 = -a\hat{B}_0\left(\frac{2\mu}{(\mu+\mu_0)G}\right)$$

Note: if $a=b$,

$c_1 = 0$

$f_1 = -\hat{B}_0 a$

$\Rightarrow$ influence of μ drops out

Note: if $\mu=\mu_0$,

$G = 1$, $e_1 = -b\hat{B}_0$, $d_1 = 0$, $c_1 = 0$

$f_1 = -a\hat{B}_0$

For μ large, in the region $\rho > b$

$$\Phi_m \simeq \hat{B}_0 \cos\varphi\left[\frac{b^2}{\rho} - \rho\right]$$

$$B_\rho = -\mu_0 \frac{\partial \varphi_m}{\partial \rho} = -\cancel{\mu_0}\, B_0 \cos\varphi \left(-\frac{b^2}{\rho^2} - 1 \right)$$

$$= B_0 \cos\varphi \left(1 + \frac{b^2}{\rho^2} \right)$$

$$B_\varphi = -\mu_0 \frac{1}{\rho} \frac{\partial}{\partial \varphi} \varphi_m = B_0 \sin\varphi \left(\frac{b^2}{\rho^2} - 1 \right)$$

$\Rightarrow$ $\underset{\sim}{B}$ affected by μ over a region

$\rho \sim b$ $\Rightarrow$ B_φ reduced

Inside cylinder, $(\rho < a)$,

$$c_1 \simeq -a \hat{B}_0 \frac{2(2)\mu_0}{\mu\left(1 - \frac{a^2}{b^2}\right)}$$

$$\varphi_m \simeq -\frac{\rho}{\cancel{a}} \cancel{a} \hat{B}_0 \frac{4\mu_0}{\mu} \frac{1}{\left(1 - \frac{a^2}{b^2}\right)} \cos\varphi$$

$$= -\frac{4\mu_0}{\mu} \hat{B}_0 \frac{\rho}{1 - \frac{a^2}{b^2}} \cos\varphi$$

$\Rightarrow$ $\underset{\sim}{B}$ reduced by factor

$$\frac{4\mu_0}{\mu\left(1 - \frac{a^2}{b^2}\right)}$$

$\Rightarrow$ for $\frac{b-a}{a}$ small factor is

$$\frac{2\mu_0}{\mu(1 - a/b)}$$

Inside high μ region,

$$d_1 \simeq -\frac{b}{\sigma_1}\hat{B}_0 \frac{a^2}{b^2} \sim -\frac{b\hat{B}_0}{\frac{\mu}{2\mu_0}\left(1-\frac{a^2}{b^2}\right)}$$

$$\simeq \frac{-b\hat{B}_0}{(1-a/b)}\frac{\mu_0}{\mu}$$

$$e_1 \simeq -\frac{b\hat{B}_0}{(1-a/b)}\frac{\mu_0}{\mu}$$

$$\Phi_m = -\cos\varphi\left(+\frac{b^2}{\rho}+\rho\right)\hat{B}_0\frac{\mu_0}{\mu}\frac{1}{1-a/b}$$

$$\underset{\sim}{B} = -\mu\nabla\Phi_m$$

$$B_\varphi \simeq -(\sin\varphi)\frac{2B_0}{1-a/b} \sim B_0\frac{b}{b-a}$$

$$B_\rho \simeq 0$$

Physical argument

Before high μ material is placed, the flux F_0, per unit length in z in region of order b is

$$F_0 \sim B_0 b$$

After μ is inserted this flux is concentrated in region of high μ but F is conserved

$$F_0 = B_0 b \sim B_\mu (b-a)$$

$$\Rightarrow B_\mu \sim B_0\frac{b}{b-a} \Rightarrow \text{same as above}$$

To estimate $\underset{\sim}{B}$ just outside of high μ material, use continuity of $H_\varphi \sim \frac{B_\varphi}{\mu}$

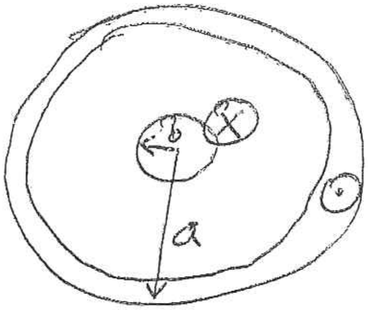
$$\frac{B_{out}}{\mu_0} \sim \frac{B_\mu}{\mu}$$

$$B_{out} \sim B_0 \frac{b}{b-a} \frac{\mu_0}{\mu} \Rightarrow \text{as before}$$

note: calculation only valid for

$$(b-a)\mu > b\mu_0$$

5.27



Calculate the inductance per unit length for currents going in the inner conductor and out the outer.

$\Rightarrow$ calculate the stored energy

$$\frac{dW_B}{dl} = \frac{1}{2}\frac{dL}{dl} I^2$$

$\underline{r > b}$

$$\nabla \times \underset{\sim}{B} = \mu_0 J$$

$$B\, 2\pi r = \mu_0 I$$

$$B = \frac{\mu_0 I}{2\pi r}$$

$\underline{r < b}$

$$B 2\pi r = \mu_0 \frac{\pi r^2}{\pi b^2} I$$

$$B = \mu_0 \frac{r}{b^2} I \frac{1}{2\pi}$$

$$\frac{dW_B}{dl} = \int_0^b 2\pi r\, dr \left(\frac{\mu_0 r I}{2\pi b^2}\right)^2 \frac{1}{2\mu_0}$$

$$+ \int_b^a 2\pi r\, dr \left(\frac{\mu_0 I}{2\pi r}\right)^2 \frac{1}{2\mu_0}$$

$$= \frac{\cancel{2\pi}}{\cancel{2\mu_0}} \frac{\mu_0^{\cancel{2}} I^2}{4\pi^{\cancel{2}} \cancel{b^4}} \frac{\cancel{b^4}}{4} + \frac{\cancel{2\pi}}{\cancel{2\mu_0}} \frac{\mu_0^{\cancel{2}} I^2}{4\pi^{\cancel{2}}} \ln\left(\frac{a}{b}\right)$$

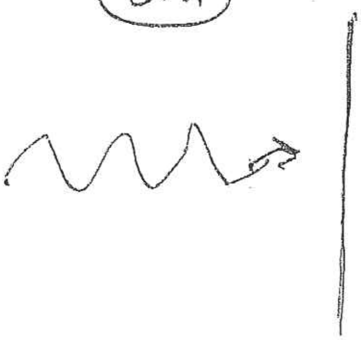
$$\frac{1}{2}\frac{dL}{dl} I^2 = \frac{\mu_0 I^2}{4\pi}\left[\frac{1}{4} + \ln\frac{a}{b}\right]$$

$$\boxed{\frac{dL}{dl} = \frac{\mu_0}{2\pi}\left(\frac{1}{4} + \ln\frac{a}{b}\right)}$$

For thin hollow tube on inside

$$\boxed{\frac{dL}{dl} = \frac{\mu_0}{2\pi}\ln\left(\frac{a}{b}\right)}$$

6.11



a) Force is momentum per unit time deposited.

$$\underset{\sim}{P}_{field} = \frac{1}{c^2}\,\underset{\sim}{E} \times \underset{\sim}{H}$$

In time Δt, momentum deposited is

$$A c \Delta t \,\frac{1}{c^2}\,\frac{EB}{\mu_0} = F\Delta t = P_{rad} A \Delta t$$

$$P_{rad} = \frac{EB}{c\mu_0} = \frac{B^2}{\mu_0} = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{B^2}{\mu_0}\right)$$

$= u =$ energy density

b)

$$a\,\frac{dm}{dA} = u \qquad uc = 1.4\,\frac{kW}{m^2}$$

$$a = \frac{1.4\times10^3\,\cancel{W}}{10^{-3}\,kg}\;\frac{Nm}{s}\;\frac{s}{3\times10^8\,m}\;\frac{kg\,m}{s^2} = 4.67\times10^{-3}\ m/s^2$$

③

From the solar wind

$$10^{-3} \frac{Kg}{m^2} a = 1.67 \times 10^{-27} \frac{Kg \ 10}{(10^{-2} m)^3} \left(\frac{400 Km}{s}\right)^2$$

$$10^{-3} \frac{\cancel{Kg}}{\cancel{m^2}} a = \frac{1.67 \times 10^{-27} (10)}{10^{-6} \cancel{m^3}} \cancel{Kg} \ 16 \times 10^4 \frac{\cancel{Km} \ 10^6 m^2}{s^2}$$

$$a = 26.7 \times 10^{-7} \frac{m}{s^2}$$

$$= 2.67 \times 10^{6} \frac{m}{s^2}$$

$\Longrightarrow$ much less than radiation pressure