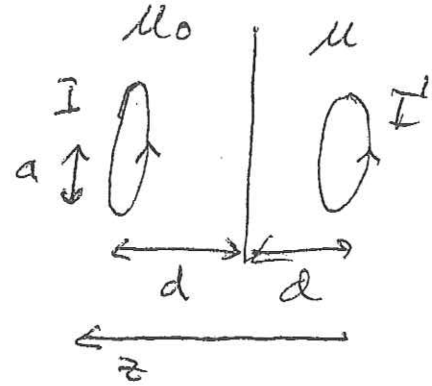
Homework 5
Solutions

5.18

a)



From 5.17 use an image current

$$I' = I\,\frac{\mu_r - 1}{\mu_r + 1}$$

$$\mu_r = \mu/\mu_0$$

Use last weeks solution of 5.10a for ~~the~~ $A_\varphi(\rho, z)$ from I'. Only I' produces a force on I $\Rightarrow$ self field does not produce a force $\Rightarrow$ force along z
$\Rightarrow$ force from B'_ρ $\Rightarrow dF_z = -I\,dl\,B'_\rho$

$$\underset{\sim}{B}' = \nabla \times A'_\varphi\,\hat{\varphi} \qquad B'_\rho = -\frac{\partial}{\partial z} A'_\varphi \qquad F_z = -2\pi a I B$$

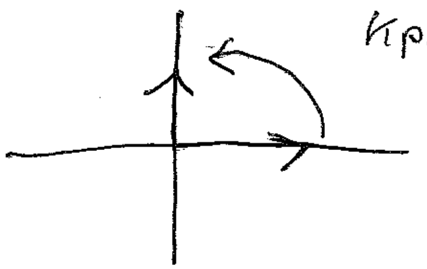
$$A'_\varphi = \frac{\mu_0 I' a}{\pi} \int_0^\infty dk \cos kz\; I_1(ka)\, K_1(ka)$$

$$F_z = -\frac{2\cancel{\pi} a\, I \mu_0 a I'}{\cancel{\pi}} \int_0^\infty dk\, k \sin(2kd)\, I_1(ka) K_1(ka)$$

$$\boxed{F_z = -2a^2 I^2 \mu_0 \frac{\mu_r - 1}{\mu_r + 1} \underbrace{\int_0^\infty dk\, k \sin(2kd)\, I_1(ka) K_1(ka)}_{G}}$$

large $d \gg a$

$$G = \mathrm{Im} \int_0^\infty dk\, k\, e^{2ikd}\, I_1(ka) K_1(ka)$$



k plane

note: $I_1(ka) K_1(ka) \sim \frac{1}{ka}$ for large ka

Rotate contour to the positive imaginary axis

define $k = is$

$dk = i\,ds$

$$G = \mathrm{Im} \int_0^\infty i\,ds\, i s\, e^{-2ds}\, I_1(isa) K_1(isa)$$

$$= -\,\mathrm{Im} \int_0^\infty ds\, s\, e^{-2ds}\, I_1(isa) K_1(isa)$$

note the integral goes to zero for $s \sim \frac{1}{d}$

$\Longrightarrow$ can expand I_1, K_1 for small argument

$$I_1(z) = \frac{z}{2}\left(1 + \frac{z^2}{8}\right), \quad K_1 = \frac{1}{z}\left(1 + \frac{z^2}{2}\ln\left(\frac{z}{2}\right) + \frac{z^2}{4}(2\gamma - 1)\right)$$

see Abramowitz and Stegun

Only the ~~la~~ term~~s~~ contributes to the imaginary part

$$G = -\int_0^\infty ds\, s\, e^{-2ds}\,\frac{1}{2}\left(\frac{-s^2a^2}{2}\right)\frac{\pi}{2}$$

$$= +\frac{a^2\pi}{8}\int_0^\infty ds\, s^3\, e^{-2ds}$$

$$= +\frac{a^2\pi}{8}\,\frac{6}{8d^4} = +\frac{3\pi}{32}\,\frac{a^2}{d^4}$$

$$\boxed{F_z = -\frac{3\pi}{16}\,\frac{a^4}{d^4}\,I^2\mu_0\,\frac{\mu_r-1}{\mu_r+1}}$$

<u>Simpler argument</u>

$$F_z = -2\pi a\, B'_\rho\, I$$

for a magnetic dipole

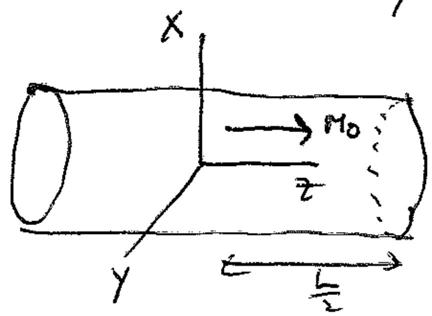
$$B'_\rho = \frac{\mu_0}{4\pi}\,\frac{3\,\hat{n}\cdot\hat{e}\,m}{(2d)^3} = \frac{\mu_0}{4\pi}\,\frac{3\,a}{(2d)^4}\,\pi a^2 I\,\frac{\mu_r-1}{\mu_r+1}$$

$$F_z = -\frac{3\pi}{32}\,\frac{a^4}{d^4}\,I^2\mu_0\,\frac{\mu_r-1}{\mu_r+1}$$

5.19a Hard cylinder of length L and radius "a" with magnetization M_0 parallel to its axis

a) Calculate $\underset{\sim}{H}$ and $\underset{\sim}{B}$ along the axis

$\Rightarrow$ only H_z, B_z by symmetry



$$\nabla \times \underset{\sim}{H} = 0$$

$$\underset{\sim}{H} = -\nabla \Phi_m$$

$$\underset{\sim}{B} = \mu_0 \left(-\nabla \Phi_m + \underset{\sim}{M} \right)$$

$$\nabla \cdot \underset{\sim}{B} = 0 \qquad \nabla^2 \Phi_m = \nabla \cdot \underset{\sim}{M} = \frac{\partial}{\partial z} M_z$$

$$= M_0 \left[\delta\left(z + \tfrac{L}{2}\right) - \delta\left(z - \tfrac{L}{2}\right) \right]$$

$\Rightarrow$ two charged discs $\qquad \otimes \; H(a - \rho)$

$$\nabla^2 \frac{1}{|x - x'|} = -4\pi \, \delta(\underset{\sim}{x} - \underset{\sim}{x}')$$

$$\rho_m \equiv \frac{M_0}{4\pi} \left[\delta\left(z - \tfrac{L}{2}\right) - \delta\left(z + \tfrac{L}{2}\right) \right] H(a - \rho)$$

$$\Phi_m = \int d\underset{\sim}{x}' \, \rho_m(\underset{\sim}{x}') \frac{1}{|\underset{\sim}{x} - \underset{\sim}{x}'|}$$

$$\Phi_m(0, \varphi, z) = 2\pi \int_0^a \int_{-\infty}^{\infty} \frac{dz' \, d\rho' \, \rho'}{\left[(z - z')^2 + \rho'^2 \right]^{\frac{1}{2}}} \frac{M_0}{4\pi} \left[\delta\left(z' - \tfrac{L}{2}\right) - \delta\left(z' + \tfrac{L}{2}\right) \right]$$

$$= \frac{M_0}{2} \int_0^a d\rho' \, \rho' \left[\frac{1}{\left[\left(z - \frac{L}{2}\right)^2 + \rho'^2 \right]^{\frac{1}{2}}} - \frac{1}{\left[\left(z + \frac{L}{2}\right)^2 + \rho'^2 \right]^{\frac{1}{2}}} \right]$$

$$\Phi_m(0,\rho,z) = \frac{M_0}{2}\left[\left[\left(z-\frac{L}{2}\right)^2+\rho'^2\right]^{\frac{1}{2}} - \left[\left(z+\frac{L}{2}\right)^2+\rho'^2\right]^{\frac{1}{2}}\right]_0^a$$

$$= \frac{M_0}{2}\left[\left[\left(z-\frac{L}{2}\right)^2+a^2\right]^{\frac{1}{2}} - \left[\left(z+\frac{L}{2}\right)^2+a^2\right]^{\frac{1}{2}} - \left|z-\frac{L}{2}\right| + \left|z+\frac{L}{2}\right|\right]$$

$$H_z = -\frac{\partial \Phi_m}{\partial z}$$

$\underline{z > \frac{L}{2}}$

$$H_z = -\frac{M_0}{2}\left[\frac{z-\frac{L}{2}}{\left[\left(z-\frac{L}{2}\right)^2+a^2\right]^{\frac{1}{2}}} - \frac{z+\frac{L}{2}}{\left[\left(z+\frac{L}{2}\right)^2+a^2\right]^{\frac{1}{2}}} - \cancel{1} + \cancel{1}\right]$$

$$B_z = \mu_0 H_z$$

Note for $z \gg a, \frac{L}{2}$

$$H_z = \frac{M_0 a^2 L}{2}\frac{1}{z^3} = \frac{m}{2\pi}\frac{1}{z^3}$$

with $m = M_0 \pi a^2 L$

$=$ magnetic moment

$\Rightarrow$ result from magnetic dipole

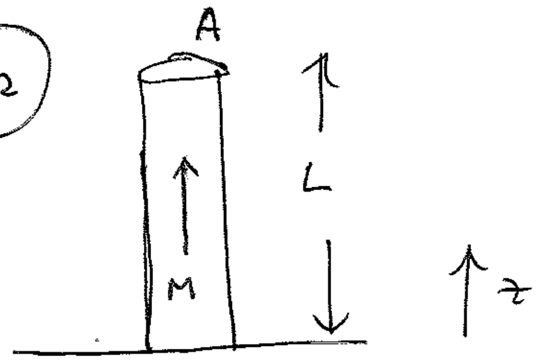
$\underline{-\frac{L}{2} < z < \frac{L}{2}}$

$$H_z = -\frac{M_0}{2}\left[\frac{z-\frac{L}{2}}{\left(\left(z-\frac{L}{2}\right)^2+a^2\right)^{\frac{1}{2}}} - \frac{z+\frac{L}{2}}{\left(\left(z+\frac{L}{2}\right)^2+a^2\right)^{\frac{1}{2}}} + 2\right]$$

$$B_z = \mu_0\left[H_z + M_0\right]$$

$\Rightarrow$ even function of z

5.22



A very long bar of magnetization M will have $B \simeq \mu_0 M$. To see this

Take $\underset{\sim}{H} = -\nabla \Phi_m$ since $\nabla \times \underset{\sim}{H} = 0$

$$\underset{\sim}{B} = \mu_0 \left(-\nabla \Phi_m + \underset{\sim}{M} \right)$$

$$\nabla \cdot \underset{\sim}{B} = 0 = \mu_0 \left[-\nabla^2 \Phi_m + \nabla \cdot \underset{\sim}{M} \right]$$

Away from the ends where $\nabla \cdot \underset{\sim}{M} \neq 0$

$$\Phi_m \simeq 0 \Rightarrow \underset{\sim}{H} \simeq 0$$

$$\Rightarrow \underset{\sim}{B} = \mu_0 \underset{\sim}{M}$$

When the bar is placed against the high μ surface B_z is continuous and

inside the surface $H_z \sim \frac{B_z}{\mu} \to 0$

Thus, inside the material $\underset{\sim}{M}$ is the same as in the magnet

Consider a small displacement of the magnetic Δz above the surface. The magnetic field in the gap is $B = \mu_0 M$ and the energy is

$$W = \frac{1}{2\mu_0} B^2 A \Delta z = \frac{1}{2} \mu_0 M^2 A \Delta z$$

$$\boxed{F_z = -\frac{\partial W}{\partial z} = -\frac{1}{2} \mu_0 M^2 A}$$

Force directly from currents

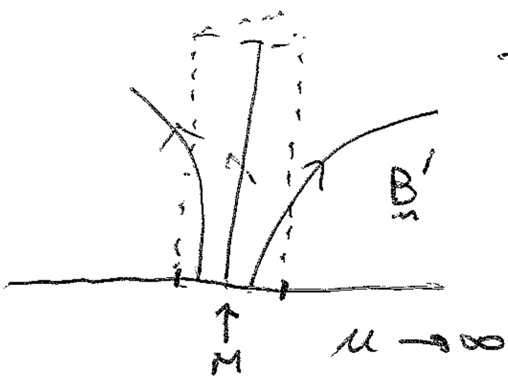
$$F_z = \int d\underset{\sim}{x} \, (\underset{\sim}{J} \times \underset{\sim}{B}')_z = -\int d\underset{\sim}{x} \, J_\varphi B_\varphi'$$

$$\underset{\sim}{J} = \nabla \times \underset{\sim}{M} \qquad J_\varphi = -\frac{\partial}{\partial e} M$$

B_e' is field from high μ medium

$$F_z = \int d\underset{\sim}{x} \, B_e' \frac{\partial}{\partial e} M = -\int d s' B_e' \, M$$

$$= -M \underbrace{\int d s' B_e}_{\text{total flux through surface of cylinder}}$$



Flux through cylindrical walls equals that through bottom

$$\Rightarrow B'A = \frac{\mu_0 M A}{2}$$

$$F_z = -\frac{\mu_0 M^2 A}{2}$$

$\Rightarrow$ note $\frac{1}{2}$ arises because at the interface half of the magnetic field is from cylinder and half is from surface

(see prob. 5.19 at $z = \frac{L}{2}$)