②

Hwk #4 Solutions

(5.8)

Eqn for A_φ

$$\nabla^2 A_\varphi - \frac{1}{r^2 \sin^2\theta} A_\varphi = -\mu_0 J_\varphi(r,\theta)$$

⇒ calculate the response to a unit current

$$j = \frac{\delta(r-r')}{r}\,\delta(\theta-\theta')$$

$$\int_{-\pi}^{\pi}\int_0^\infty dr\, r\, d\theta\, j = 1$$

$$\nabla^2 G_\ell - \frac{1}{r^2\sin^2\theta} G_\ell = -\mu_0 j$$

$$G_\ell = \sum_\ell P_\ell^1(\cos\theta)\, R_\ell(r)$$

$$\sum_\ell \frac{1}{r^2}\left(\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} R_\ell - \ell(\ell+1) R_\ell\right) P_\ell^1 = -\mu_0 j$$

⇒ eliminate the sum over ℓ by multiplying by $P_{\ell'}^1(\cos\theta)$ and integrating over $d\cos\theta$

$$\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} R_\ell - \ell(\ell+1) R_\ell = -\mu_0 r \int_{-1}^{1} d\cos\theta\, \frac{\delta(r-r')}{\cancel{r}}\,\delta(\theta-\theta') \otimes P_\ell^1(\cos\theta)\, \frac{2\ell+1}{2}\, \frac{1}{\ell(\ell+1)}$$

$$\int_{-1}^{1} d\cos\theta\, (P_\ell^1)^2 = \frac{2}{2\ell+1}\frac{(\ell+1)!}{(\ell-1)!}$$

$$= -\mu_0 r\, \delta(r-r')\, P_\ell^1(\cos\theta') \sin\theta' \otimes \frac{2\ell+1}{2}\, \frac{1}{\ell(\ell+1)}$$

Let $S_\ell = -\mu_0 \, r' \, P_\ell^1(\cos\theta') \sin\theta' \, \frac{2\ell+1}{2\ell(\ell+1)}$

$$\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} R_\ell - \ell(\ell+1) R_\ell = S_\ell \, \delta(r-r')$$

<u>$r \neq r'$</u>

$$R_\ell = \left(\frac{r}{r'}\right)^\ell C_\ell \qquad r<r'$$

$$= \left(\frac{r'}{r}\right)^{\ell+1} C_\ell \qquad r>r'$$

$\Rightarrow$ continuity at $r=r'$

<u>$r \simeq r'$</u>

$$\frac{\partial^2}{\partial r^2} R_\ell \simeq \frac{1}{r^2} S_\ell \, \delta(r-r')$$

$$\left.\frac{\partial R_\ell}{\partial r}\right|_{r'-\epsilon}^{r'+\epsilon} = \frac{1}{r'^2} S_\ell$$

$$(-2\ell-1) C_\ell \frac{1}{r'} = \frac{1}{r'^2} S_\ell$$

$$C_\ell = -\frac{S_\ell}{2\ell+1} \frac{1}{r'}$$

$$G_\ell = \frac{\mu_0}{2} \sin\theta' \sum_\ell \frac{1}{\ell(\ell+1)} P_\ell^1(\cos\theta') P_\ell^1(\cos\theta) \otimes \frac{r_<^\ell}{r_>^{\ell+1}} r'$$

<u>interior</u>

$$A_\varphi = \int_0 \int_{-\pi}^{\pi} dr' \, r' \, d\theta' \, G(\;) \, J_\varphi(r', \theta')$$

$$A_\varphi = \frac{\mu_0}{2}\sum_\ell \underbrace{\int dr'\, d\cos\theta'\, r'^2}_{\frac{1}{2\pi}\int d\underline{x}'} \frac{\cancel{1}}{\ell(\ell+1)} \frac{P_\ell^1(\cos\theta')\, P_\ell^1(\cos\theta)}{r'^{\ell+1}} J_\varphi$$

$$A_\varphi = -\frac{\mu_0}{4\pi}\sum_\ell m_\ell\, r^\ell\, P_\ell^1(\cos\theta)$$

$$m_\ell = -\int d\underline{x}' \frac{1}{\ell(\ell+1)} P_\ell^1(\cos\theta') \frac{1}{r'^{\ell+1}} J_\varphi(r',\theta')$$

<u>outside</u>

$$A_\varphi = -\frac{\mu_0}{4\pi}\sum_\ell \mathcal{M}_\ell \frac{P_\ell^1(\cos\theta)}{r^{\ell+1}}$$

$$\mathcal{M}_\ell = -\frac{1}{\ell(\ell+1)}\int d\underline{x}'\, r'^\ell\, P_\ell^1(\cos\theta')\, J_\varphi(r',\theta')$$

5.10

a) $\nabla^2 A_\varphi - \frac{1}{\rho^2} A_\varphi = -\mu_0 J_\varphi(\rho, z)$ 1

$J_\varphi = I \; \delta(\rho - a)\delta(z)$

note $\int_0^\infty \int_{-\infty}^{\infty} dz\, d\rho\, J_\varphi = I$

Let $A_\varphi = \int_0^\infty dk \cos kz \, R_k(\rho)$

$\Rightarrow$ even in z and oscillatory

$\Rightarrow$ non-oscillatory in ρ

$\Rightarrow$ continuous since ∞ system

insert into Eq (1) $\Rightarrow$

$$\int_0^\infty dk \cos kz \left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} - k^2 - \frac{1}{\rho^2}\right] R_k = -\mu_0 J_\varphi$$

$\Rightarrow$ eliminate k integral

$\Rightarrow$ operate with $\int dz \cos k'z$

$$\int dz \cos(k'z)\cos kz = \pi \int \frac{dz}{4\pi}\left(e^{ik'z} + e^{-ik'z}\right)\left(e^{ikz} + e^{-ikz}\right)$$

$$= \frac{\pi}{2}\left[2\delta(k-k') + 2\,\cancel{\delta(k+k')}\right] = \pi\,\delta(k-k')$$

$$\pi\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} - k^2 - \frac{1}{\rho^2}\right]R_k = -\mu_0 I\,\delta(\rho - a)$$

$$\left[\rho\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} - k^2\rho^2 - 1\right]R_k = -\frac{\mu_0}{\pi} a^2 I\,\delta(\rho - a)$$

<u>$\rho \neq a$</u> $\left(\rho \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} - k^2 \rho^2 - 1\right) R_k = 0$

$\Rightarrow$ Modified Bessel Eqn of order 1.

$R_k \sim I_1(k\rho),\ K_1(k\rho)$

$$R_k = c_k\, I_1(k\rho_<)\, K_1(k\rho_>)$$

$\Rightarrow$ I_1 bounded as $k\rho \to 0$

K_1 bounded as $k\rho \to \infty$

$\rho_<$ = smaller of ρ, a

$\rho_>$ = larger of ρ, a

<u>$\rho \simeq a$</u>

$$\frac{\partial^2}{\partial \rho^2} R_k = -\frac{\mu_0}{\pi} \cancel{a^2} I\, \delta(\rho - a)$$

$$\frac{\partial}{\partial \rho} R_k \Big|_{\rho-\epsilon}^{\rho+\epsilon} = -\frac{\mu_0}{\pi} I$$

$$k\, c_k \left[K_1'(ka) I_1(ka) - I_1'(ka) K_1(ka) \right] = -\frac{\mu_0}{\pi} I$$

$$c_k = -\frac{\mu_0}{\pi} I a \frac{1}{W(I_1, K_1)} \frac{1}{ka}$$

$$= \frac{\mu_0}{\pi} I a$$

$$A_\varphi = \frac{\mu_0 I a}{\pi} \int_0^\infty dk \cos kz\, I_1(k\rho_<) K_1(k\rho_>)$$

c) $\underset{\sim}{B} = \nabla \times (A_\phi \hat{\phi})$

$$\hat{\phi} = \rho \frac{\hat{\phi}}{\rho} = \rho \nabla \phi$$

$$\underset{\sim}{B} = \nabla \times (A_\phi \rho \nabla \phi) = \nabla(\rho A_\phi) \times \nabla \phi$$

$$B_z = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho A_\phi)$$

$$B_\phi = 0$$

$$B_\rho = -\frac{1}{\rho} \frac{\partial}{\partial z} \rho A_\phi = -\frac{\partial}{\partial z} A_\phi$$

<u>Limit $z = 0$ and $\rho = a - \epsilon$ with ϵ small</u>

$\Rightarrow$ should find $B_z \sim \frac{1}{\epsilon}$ $\Rightarrow$ like infinite wire

$B_\rho = 0$

Since $B_\rho = -\frac{\partial}{\partial z} A_\phi \sim \sin(kz) \longrightarrow 0$

$\Rightarrow B_\rho = 0$

$$B_z = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho A_\phi) = \frac{\mu_0 I a}{\pi} \int_0^\infty dk \, \frac{1}{\rho} \underbrace{\frac{\partial}{\partial \rho} [\rho \, I_1(k\rho)]} K_1(ka)$$

but $\frac{d}{dx} [x I_1(x)] = x I_0(x)$

$\frac{d}{dx}(x I_1) = x I_0$
$= k\rho I_0$

$$= \frac{\mu_0 I a}{\pi} \int_0^\infty dk \, k \, I_0(k\rho) K_1(ka)$$

Note that the integral diverges as $\rho \to a$ because for large k,

$$I_0 \sim \frac{e^{k\rho}}{(k\rho)^{1/2}} \text{ and } K_1 \sim \frac{e^{-ka}}{(ka)^{1/2}}$$

so $k I_0(k\rho) K_1 \sim e^{-k(a-\rho)}$

$\Rightarrow$ integrand dominated by large k

$\Rightarrow$ expand I_0, K_1 for large argument

$$I_0 = \frac{e^{x}}{\sqrt{2\pi x}}, \quad K_1 = \sqrt{\frac{\pi}{2x}}\, e^{-x}$$

$$B_z = \frac{\mu_0 I a}{\pi} \int_0^\infty dk\, k \frac{e^{k\rho}}{\sqrt{2\pi k\rho}} \sqrt{\frac{\pi}{2ka}}\, e^{-ka}$$

$$= \frac{\mu_0 I a}{2\pi a} \int_0^\infty dk\, e^{-k(a-\rho)}$$

$$= \frac{\mu_0 I}{2\pi} \frac{1}{a-\rho} \quad \Rightarrow \text{like infinite wire}$$