Physics 611 Homework #3 Spring '22

Dr. Drake

1. Jackson 4.8
2. Jackson 4.9
3. Jackson 4.13
4. Consider a square capacitor ($L \times L$) with plate separation d with $d \ll L$ and total charge Q. A block of dielectric of dielectric constant ϵ, mass m and dimensions equal to the interior of the capacitor is placed just outside of the capacitor. At $t = 0$ the dielectric is moved just inside the capacitor plates with zero initial velocity and then released. Neglect the fringing of the capacitor field and any friction between the capacitor plates and the dielectric.

   (a) Describe what happens to the block of dielectric (the capacitor is held fixed).

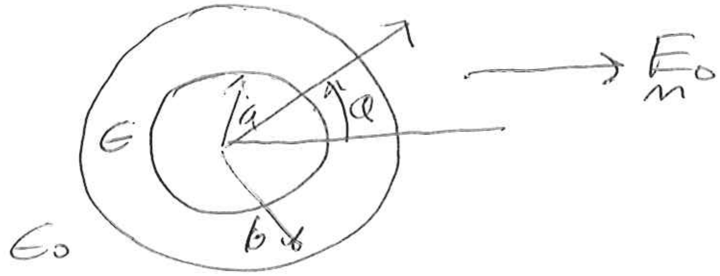
   (b) Derive an equation of motion for the dielectric.

   (c) Derive an expression for the velocity when the dielectric is completely inside the capacitor.

   (d) Derive an expression for the period τ of the motion of the dielectric. Evaluate τ for $\epsilon = \epsilon_0(1 + \delta)$ with δ small and for $\delta \gg 1$. Describe the physics in those limits.

Hwk #3 Solutions

4.8 Long, cylindrical shell of dielectric constant ϵ



$r > b$

$$\Phi = \sum_n a_n \left(\frac{b}{\rho}\right)^n \cos n\varphi - E_0 \rho \cos\varphi$$

$\Rightarrow$ even in φ
$\Rightarrow$ symmetry

$a < r < b$

$$\Phi = \sum_n \left[b_n \left(\frac{\rho}{b}\right)^n + c_n \left(\frac{b}{\rho}\right)^n \right] \cos n\varphi$$

$0 < r < a$

$$\Phi = \sum_n d_n \left(\frac{\rho}{a}\right)^n \left[b_n \left(\frac{a}{b}\right)^n + c_n \left(\frac{b}{a}\right)^n \right] \cos n\varphi$$

For $n \neq 1$ have 2 BCs at "a" and "b"
$\Rightarrow$ over constrained
$\Rightarrow a_n, b_n, c_n, d_n = 0$

For $n = 1$

radial $\underset{\sim}{D}$

at b: ① $\epsilon_0\left(-a_1 \frac{1}{b} - E_0\right) = \left(\frac{b_1}{b} - c_1 \frac{1}{b}\right)\epsilon$

at a: ② $d_1 \frac{E_0}{a}\left[b_1 \frac{a}{b} + c_1 \frac{b}{a}\right] = \left(b_1 \frac{1}{b} - c_1 \frac{b}{a^2}\right)\epsilon$

Azimuthal E

at b: (3) $a_1 - E_0 b = b_1 + c_1$

at a: $b_1\left(\frac{a}{b}\right) + c_1\left(\frac{b}{a}\right) = d_1\left[b_1\left(\frac{a}{b}\right) + c_1\frac{b}{a}\right]$

$$\Rightarrow d_1 = 1$$

Eqn (2) becomes

$$b_1\left(\frac{\epsilon_0}{b} - \frac{\epsilon}{b}\right) = -c_1\frac{b}{a^2}(\epsilon + \epsilon_0)$$

$$\frac{b_1}{\cancel{b}}(\epsilon - \epsilon_0) = c_1\frac{b^2}{a^2}(\epsilon + \epsilon_0)$$

$$c_1 = \frac{a^2}{b^2}\,\frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0}\,b_1$$

Eq (3) becomes

(4) $$a_1 = E_0 b + b_1\left[1 + \frac{a^2}{b^2}\,\frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0}\right]$$

Eq. (1) becomes

(5) $$a_1 + E_0 b = -\frac{\epsilon}{\epsilon_0}\left(1 + \frac{a^2}{b^2}\,\frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0}\right)b_1$$

subtract (5) from (4). Let $h \equiv \frac{a^2}{b^2}\,\frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0}$

$$-2E_0 b = b_1\left[1 + h + \frac{\epsilon}{\epsilon_0}(1 + h)\right]$$

$$\cancel{-2\epsilon_0 E_0 b = b_1[\epsilon + \epsilon_0 + h(\epsilon + \epsilon_0)]}$$

$$\cancel{= b_1\left[1 + \frac{a^2}{b^2}\right](\epsilon + \epsilon_0)}$$

$$b_1 = -2E_0 b \frac{\epsilon_0}{\epsilon+\epsilon_0} \frac{1}{1+\frac{a^2}{b^2}\frac{\epsilon-\epsilon_0}{\epsilon+\epsilon_0}}$$

~~note~~ $$c_1 = \frac{a^2}{b^2}\frac{\epsilon-\epsilon_0}{\epsilon+\epsilon_0} b_1$$

Add (4) and (5)

$$2a_1 = b_1 (1+h)\left(1-\frac{\epsilon}{\epsilon_0}\right)$$

$$a_1 = +\frac{1}{\cancel{2}}\left(\frac{\epsilon-\epsilon_0}{\epsilon_0}\right)\left(+\frac{\cancel{2}E_0 b\,\epsilon_0}{\epsilon+\epsilon_0}\right)$$

$$a_1 = \frac{\epsilon-\epsilon_0}{\epsilon+\epsilon_0} E_0 b$$

note for $\epsilon = \epsilon_0 \Rightarrow b_1 = -E_0 b$

$\cancel{\epsilon_0}$ $c_1 = a_1 = 0$

$\Rightarrow$ uniform field

<u>$r > b$</u>

$$\Phi = a_1 \frac{b}{\rho}\cos\varphi - E_0 \rho \cos\varphi$$

<u>$a < r < b$</u>

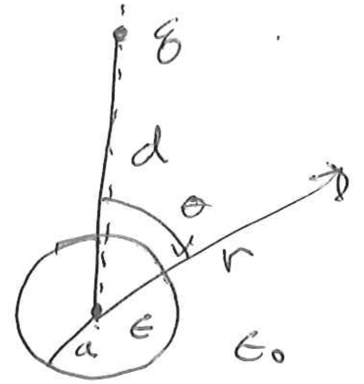
$$\Phi = \left(b_1 \frac{\rho}{b} + c_1 \frac{b}{\rho}\right)\cos\varphi$$

<u>$r < a$</u>

$$\Phi = \frac{\rho}{a}\left[b_1 \frac{a}{b} + c_1 \frac{b}{a}\right]\cos\varphi$$

4.9

a)



You could solve this by expanding in Spherical harmonics for $r > d$, $a < r < d$ and $r < a$ and doing the jump conditions at $r = d$ and $r = a$ but since you know the solution for q in the absense of the sphere, it is easier to write the potential in terms of that due to q, Φ_q, and that due to the sphere, Φ_ϵ. Then you only ~~had~~ have to do matching at $r = a$.

change q

$$\Phi_q = \frac{q}{4\pi\epsilon_0} \sum_\ell \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos\theta)$$

$\Rightarrow$ note $\frac{\partial}{\partial\varphi} = 0$

$r_<$ = smaller of r, d
$r_>$ = larger of r, d

dielectric ϵ

$r > a$
$$\Phi_\epsilon = \frac{q}{4\pi\epsilon_0} \sum_\ell a_\ell \left(\frac{a}{r}\right)^{\ell+1} P_\ell(\cos\theta)$$

$r < a$
$$\Phi_\epsilon = \frac{q}{4\pi\epsilon_0} \sum_\ell b_\ell \left(\frac{r}{a}\right)^{\ell} P_\ell(\cos\theta)$$

BCs at $r=a$

continuity of E_θ $\Rightarrow$ ~~Φ_q~~ Φ_q already continuous

$$-\frac{q}{4\pi\epsilon_0}\sum_\ell a_\ell\, P_\ell'(\cos\theta)\sin\theta$$

$$= -\frac{q}{4\pi\epsilon_0}\sum_\ell b_\ell\, P_\ell'(\cos\theta)\sin\theta$$

note: $\left[\frac{d}{d\cos\theta} P_\ell(\cos\theta)\right]\sin\theta \sim P_\ell^1(\cos\theta)$

$\Rightarrow$ orthogonal

$\Rightarrow$ can reduce series to single P_ℓ

$$\boxed{a_\ell = b_\ell}$$

continuity of D_r : $\Phi_q = \frac{q}{4\pi\epsilon_0}\sum_\ell \frac{r^\ell}{d^{\ell+1}} P_\ell$

$\Rightarrow$ project to single ℓ $\Rightarrow$ P_ℓ orthogonal

$$\epsilon_0\left[\ell\frac{a^{\ell-1}}{d^{\ell+1}} - a_\ell(\ell+1)\frac{1}{a}\right]$$

$$= \epsilon\left[\ell\frac{a^{\ell-1}}{d^{\ell+1}} + \ell\frac{b_\ell}{a}\right]$$

$$a_\ell\left(\ell\epsilon + (\ell+1)\epsilon_0\right) = -\frac{\ell a^\ell}{d^{\ell+1}}(\epsilon-\epsilon_0)$$

$$b_\ell = a_\ell = -\frac{a^\ell}{d^{\ell+1}} \frac{\ell(\epsilon-\epsilon_0)}{\ell\epsilon+(\ell+1)\epsilon_0}$$

b) near center of sphere

$$\Phi = \frac{q}{4\pi\epsilon_0} \sum_\ell \left[\frac{r^\ell}{d^{\ell+1}} + b_\ell \frac{r^\ell}{a^\ell}\right] P_\ell(\cos\theta)$$

$\Rightarrow$ only $\ell=1$ contributes as $r\to 0$

$$\Phi = \frac{q}{4\pi\epsilon_0} z \left[\frac{1}{d^2} + \frac{b_1}{a}\right]$$

$$b_1 = -\frac{a}{d^2} \frac{(\epsilon-\epsilon_0)}{\epsilon+2\epsilon_0}$$

$$\Phi = \frac{q}{4\pi\epsilon_0} z \left[\frac{1}{d^2} - \frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right]$$

$$E_z = -\frac{q}{4\pi\epsilon_0}\frac{1}{d^2}\left[1 - \frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right]$$

$$= -\frac{q}{4\pi\epsilon_0 d^2} \frac{3\epsilon_0}{\epsilon+2\epsilon_0}$$

c) As $\epsilon \to \infty$

$E_z \to 0$

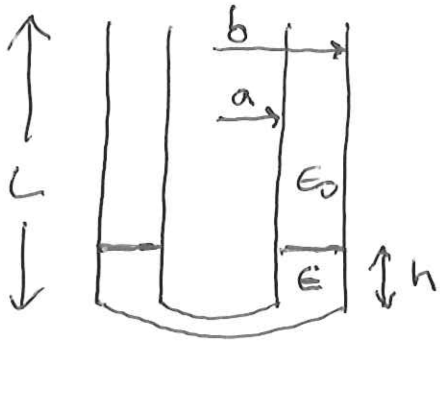
4.13

Have a long cylindrical capacitor of inner radius "a" and outer radius "b". Applied voltage V. How far up will the fluid of dielectric constant ϵ move?

⇒ this problem can be solved by either assuming V is a constant and finding the variation of the stored energy with height. This requires including the variation of Q with height to maintain V and therefore the energy supplied by the battery (- QV). See the solution of 1.9.

⇒ easier approach is to take Q to be constant as the fluid height varies. The results must be the same as in 1.9.

The capacitor has a height L with the fluid a height h. Consider a capacitor with charge/length λ and dielectric constant ϵ. The electric field is



$$E = \frac{\lambda}{2\pi\epsilon r}$$

Since the potential drop is the same in the fluid and vacuum,

$$\frac{\lambda_0}{\epsilon_0} = \frac{\lambda_\epsilon}{\epsilon}$$

with λ_0 and λ_ϵ, the charge/length in vacuum and fluid. The energy per unit length stored in the capacitor is

$$w = \frac{1}{2}\epsilon\int_a^b dr\, 2\pi r E^2 = \frac{1}{2}\epsilon \frac{\lambda^2}{4\pi^2\epsilon^2} 2\pi \int_a^b dr \frac{1}{r}$$

$$= \frac{\lambda^2}{4\pi\epsilon} \ln\left(\frac{b}{a}\right)$$

The total energy w_{tot} is given by

$$w_{TOT} = \frac{\ln\left(\frac{b}{a}\right)}{4\pi}\left[\frac{\lambda_0^2}{\epsilon_0}(L-h) + \frac{\lambda_\epsilon^2}{\epsilon} h\right]$$

~~In terms of~~

$$= \frac{\ln\left(\frac{b}{a}\right)}{4\pi}\frac{\lambda_0^2}{\epsilon_0}\left[L-h+\frac{\epsilon}{\epsilon_0}h\right]$$

In terms of the charge $Q = \lambda_0(L-h) + \lambda_\epsilon h$

$$= \lambda_0\left[L-h+\frac{\epsilon}{\epsilon_0}h\right]$$

$$w_{TOT} = \frac{\ln\left(\frac{b}{a}\right)}{4\pi}\frac{Q^2}{\epsilon_0}\frac{1}{L-h+\frac{\epsilon}{\epsilon_0}h}$$

For $L \gg h$, $\quad w_{TOT} \simeq \frac{\ln\left(\frac{b}{a}\right)}{4\pi\epsilon_0 L} Q^2\left[1-\left(\frac{\epsilon}{\epsilon_0}-1\right)\frac{h}{L}\right]$

$\Rightarrow$ w_{TOT} decreases as h increases. $\Rightarrow$ upward force

$$F_{em} = -\frac{\partial w_{TOT}}{\partial h} = \frac{\ln\left(\frac{b}{a}\right)Q^2}{4\pi\epsilon_0 L^2}\left(\frac{\epsilon}{\epsilon_0}-1\right)$$

Gravitation force

$$F_g = \pi(b^2-a^2)\varrho g h$$

Take $\epsilon = \epsilon_0(1+\chi_e)$

$$\pi(b^2-a^2)\varrho g h = \frac{\ln\left(\frac{b}{a}\right)}{4\pi\epsilon_0}\frac{Q^2}{L^2}\chi_e$$

$$V = \int_a^b dr\, E = \frac{\lambda_0}{2\pi\epsilon_0}\ln\left(\frac{b}{a}\right)$$

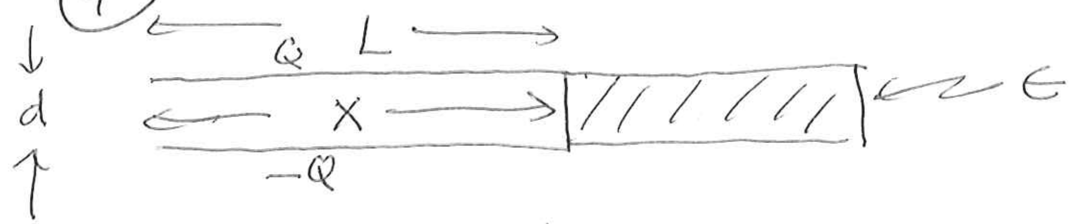
$$\simeq \frac{Q}{2\pi\epsilon_0 L}\ln\left(\frac{b}{a}\right) \quad \text{for } L \gg h$$

$$\cancel{\pi}(b^2-a^2)\varrho g h = \frac{\cancel{\ln\left(\frac{b}{a}\right)}}{\cancel{4\pi\epsilon_0}}\frac{\cancel{4\pi^2}\epsilon_0^{\cancel{2}} V^2}{\left[\ln\left(\frac{b}{a}\right)\right]^{\cancel{2}}}\chi_e$$

$$\Rightarrow \quad \chi_e = \frac{(b^2-a^2)\varrho g h \ln\left(\frac{b}{a}\right)}{\epsilon_0 V^2}$$

a) The dielectric is pulled into the capacitor and oscillates periodically (9)

(4)



b) Find stored energy for $0<X<L$.
Charge moves so E is a constant along the capacitor $\Rightarrow$ σ/ϵ is constant

$$\Rightarrow \sigma_{\epsilon_0},\ \sigma_{\epsilon}$$

$$\sigma_{\epsilon_0} XL + \sigma_{\epsilon} L(L-X) = Q = \sigma_0 L^2$$

$$\frac{\sigma_{\epsilon_0}}{\epsilon_0} = \frac{\sigma_{\epsilon}}{\epsilon}$$

$$\sigma_{\epsilon_0}\left[XL + \frac{\epsilon}{\epsilon_0}(L-X)L\right] = \sigma_0 L^2 = Q$$

$$W_{TOT} = \frac{1}{2}\int d\underset{\sim}{x}\, \epsilon E^2 \qquad E = \frac{\sigma_{\epsilon_0}}{\epsilon_0}$$

$$= \frac{1}{2} E^2 \left[\epsilon(L-X) + \epsilon_0 X\right] L d$$

$$= \frac{1}{2} Q^2 \frac{Ld}{\left[\epsilon(L-X)+\epsilon_0 X\right] L^2}$$

$$W_{TOT} = \frac{1}{2} Q^2 \frac{d}{L} \frac{1}{\epsilon(L-X)+\epsilon_0 X}$$

$$F = -\frac{\partial W_{TOT}}{\partial X} = -\frac{1}{2} Q^2 \frac{d}{L} \frac{(\epsilon-\epsilon_0)}{\left[\epsilon(L-X)+\epsilon_0 X\right]^2}$$

$$m\ddot{X} = -\frac{1}{2} Q^2 \frac{d}{L} \frac{\epsilon-\epsilon_0}{\left[\epsilon(L-X)+\epsilon_0 X\right]^2}$$

c)

Use energy conservation

ΔW = change in electric field energy

$$= W_{TOT}(x=0) - W_{TOT}(x=L)$$

$$= \frac{1}{2} Q^2 \frac{d}{L}\left[\frac{1}{\epsilon L} - \frac{1}{\epsilon_0 L}\right]$$

$$= \frac{1}{2} Q^2 \frac{d}{L^2} \frac{\epsilon_0 - \epsilon}{\epsilon_0 \epsilon}$$

$$\frac{1}{2} m v^2 = -\Delta W$$

$$v = \sqrt{\frac{Q^2}{m} \frac{d}{L^2} \frac{\epsilon - \epsilon_0}{\epsilon_0 \epsilon}}$$

d)

$$\frac{1}{2} m v^2 = W_{TOT}(x=L) - W_{TOT}(x)$$

$$= \frac{1}{2} Q^2 \frac{d}{L}\left[\frac{1}{\epsilon_0 L} - \frac{1}{\epsilon(L-x) + \epsilon_0 x}\right]$$

$$\tau = 4 \int_L^0 \frac{d\tau}{dx}\, dx = 4 \int_L^0 \frac{dx}{v}$$

For $\epsilon = \epsilon_0(1+\delta)$

For $\delta \ll 1$, easier to go back to equation of motion

$$m\ddot{x} \simeq -\frac{1}{2} Q^2 \frac{d}{L} \frac{\epsilon_0 \delta}{\epsilon_0^2 L^2}$$

$$\Leftrightarrow \ddot{x} = -\frac{1}{2} \frac{Q^2}{m} \frac{d}{\epsilon_0 L^3} \delta$$

$$= \text{const.}$$

Let $a_x = \frac{1}{2} \frac{Q^2}{m} \frac{d}{\epsilon_0 L^3} \delta$

To go a distance $x = L$

$$L = \frac{1}{2} a_x t^2 \quad \Rightarrow \tau = 4t$$

$$\tau = 4 \sqrt{\frac{2L}{a_x}}$$

$$= 4 \sqrt{\frac{2 L \epsilon_0 L^3 m}{\frac{1}{2} Q^2 d \delta}}$$

$$\boxed{\tau = \frac{8L^2}{Q} \sqrt{\frac{\epsilon_0 m}{d \delta}}}$$

For $\delta \gg 1$, W_{TOT} quickly drops to zero so the dielectric has a constant velocity

$$V \simeq \frac{Q}{L} \sqrt{\frac{d}{m\epsilon_0}}$$

$$\Rightarrow Vt = L \qquad \tau = 4t$$

$$\boxed{\tau = 4 \frac{L^2}{Q} \sqrt{\frac{m\epsilon_0}{d}}}$$