Physics 611 Homework #2 Spring '22

Dr. Drake

1. Jackson 3.20 (In (a) calculate the answer from first principles. You don't need to connect your answers to other problems in (b) and (c).)

2. Consider an infinite cylindrical conductor of radius "a" which is sliced at $z = \pm L/2$. The segment $|z| < L/2$ is maintained at a potential V. The remainder of the cylinder is grounded.

   (a) Sketch the electric field E in the region $\rho = \sqrt{x^2 + y^2} > a$.

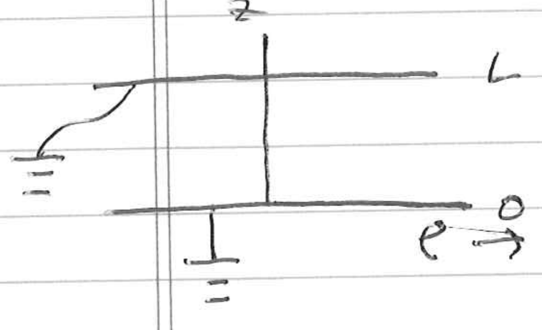
   (b) Consider the limit in which $a >> L$. Sketch the electric field in the region $\rho > a$ in this limit. Estimate the magnitude of the electric field at the surface in the region $|z| \sim 0$ and the force per unit area acting on the conductor in this region. What is the direction of the force? Over what scale length does the potential fall off in the radial and axial directions?

   (c) Derive an exact expression for the potential Φ in the region $\rho > a$ for arbitrary L/a. The solution takes the form of an integral.

   (d) Now take the limiting case where $a \gg L$. What is the characteristic scale length remaining in the problem? What physical system does the solution represent? Explicitly evaluate the radial electric field just outside of the cylinder for $z \sim 0$. The integration in this case can be completed. Check your previous estimate of E from part (b).

3.20 Take the charge to be at radius ϵ and then let $\epsilon \to 0$

$$\nabla^2 \Phi = -\frac{1}{\epsilon_0} q \frac{\delta(\rho-\epsilon)}{2\pi\rho}\delta(z-z_0)$$



Choose basis functions $\sin\left(\frac{n\pi z}{L}\right)$ in z.

$$\Phi = \sum_n R_n(\rho)\sin\left(\frac{n\pi z}{L}\right)$$

$$\sum_n \left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}R_n - \frac{n^2\pi^2}{L^2}R_n\right]\sin\frac{n\pi z}{L} = -\frac{1}{\epsilon_0}q\frac{\delta(\rho-\epsilon)}{2\pi\rho}$$

$$\times\ \delta(z-z_0)$$

mult by $\sin\left(\frac{n'\pi z}{L}\right)$ and integrate $(0, L)$

$\Rightarrow$ eliminates sum over n

$$\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}R_n - \frac{n^2\pi^2}{L^2}R_n\right)\frac{1}{2}L = -\frac{q}{\epsilon_0}\frac{\delta(\rho-\epsilon)}{2\pi\rho}\sin\frac{n\pi z_0}{L}$$

$\Rightarrow$ $\nu = 0$ Bessel's Eqn

jump conditions

$$\epsilon\,\cancel{\rho}\frac{\partial}{\partial\rho}R_n\Big|_{\epsilon_-}^{\epsilon_+} = -\frac{2q}{L\epsilon_0}\,\cancel{q}\,\frac{1}{2\pi\cancel{\rho}}\sin\left(\frac{n\pi z_0}{L}\right)$$

$$R_n\Big|_{\epsilon_-}^{\epsilon_+} = 0$$

$\rho > \epsilon_0$:

bounded solution as $\rho \to \infty$

$$R_n = c_n I_0(k_n \epsilon) K_0(k_n \rho)$$

$\rho \le \epsilon_0$

bounded solution as $\rho \to 0$

$$R_n = c_n I_0(k_n \rho) K_0(k_n \epsilon)$$

$$k_n \epsilon c_n \left[I_0(k_n \epsilon) K_0'(k_n \epsilon) - I_0'(k_n \epsilon) K_0(k_n \epsilon) \right]$$

$$= -\frac{2q}{L\epsilon_0} \frac{1}{2\pi} \sin\left(\frac{n\pi z_0}{L}\right)$$

$$k_n \epsilon c_n \left[\frac{1}{\Gamma(1)} \left(-\frac{1}{k_n \epsilon}\right) - 0 \right]$$

$$c_n = \frac{\cancel{2} q}{L\epsilon_0} \frac{1}{\cancel{2}\pi} \sin\left(\frac{n\pi z_0}{L}\right)$$

$$\Phi = \frac{q}{\pi \epsilon_0 L} \sum_n \sin\left(\frac{n\pi z_0}{L}\right) \sin\left(\frac{n\pi z}{L}\right) K_0\left(\frac{n\pi \rho}{L}\right)$$

(b) <u>lower plane</u> $E_z = -\frac{\partial \Phi}{\partial z}\Big|_{z=0}$ $\qquad \sigma_0 = \epsilon_0 E_z$

$$E_z = -\frac{q}{\cancel{\pi}\epsilon_0 L}\sum_n \frac{n\cancel{\pi}}{L}\sin\left(\frac{n\pi z_0}{L}\right) K_0\left(\frac{n\pi\rho}{L}\right)$$

$$\sigma_0 = -\frac{q}{\cancel{\pi}L^2}\sum_n n\sin\left(\frac{n\pi z_0}{L}\right) K_0\left(\frac{n\pi\rho}{L}\right)$$

<u>upper plane</u> $\qquad \sigma_L = -\epsilon_0 E_z$

$$E_z = -\frac{q}{\cancel{\pi}\epsilon_0 L}\sum_n \frac{n\cancel{\pi}}{L}\sin\left(\frac{n\pi z_0}{L}\right)(-1)^n K_0\left(\frac{n\pi\rho}{L}\right)$$

$$\sigma_L = \frac{q}{L^2}\sum_{n=1}^{\infty}(-1)^n n\sin\left(\frac{n\pi z_0}{L}\right) K_0\left(\frac{n\pi\rho}{L}\right)$$

(c) $$Q_L = \int_0^\infty d\rho\, 2\pi\rho\,\sigma_L$$

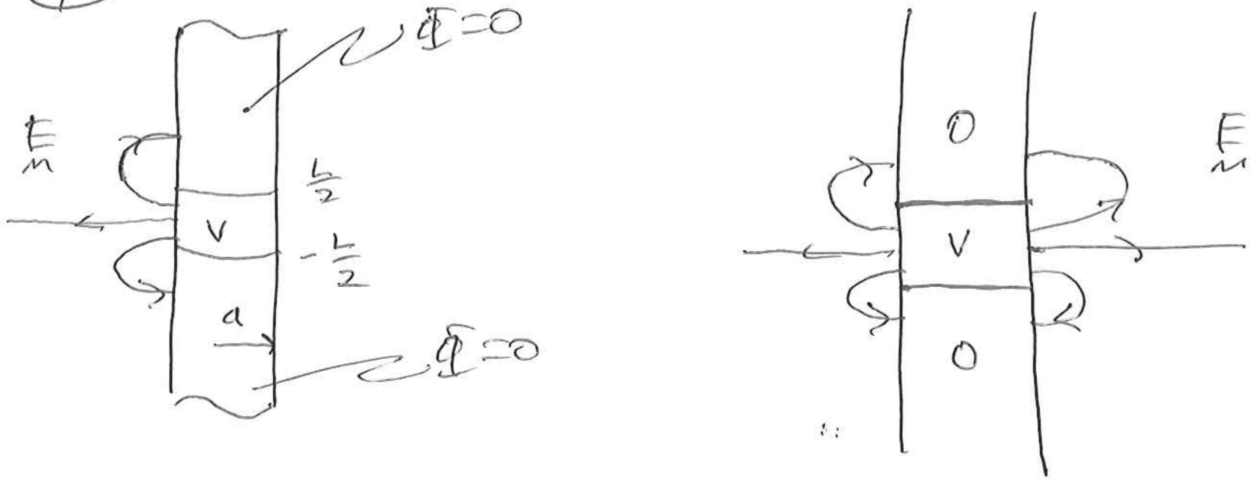
$$= \frac{2\cancel{\pi}}{\cancel{L^2}} q \sum_n \frac{(-1)^n \cancel{L^2}\, n}{n^2\pi^{\cancel{2}}}\sin\left(\frac{n\pi z_0}{L}\right)\int_0^\infty ds\, s\, K_0(s)$$

$$= \frac{2q}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\left(\frac{n\pi z_0}{L}\right)$$

The series is the discrete Fourier representation of $-z/L$ so

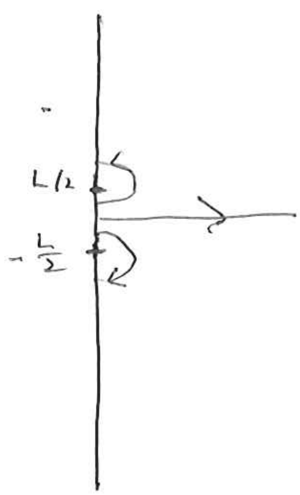
$$Q_L = -q\frac{z_0}{L}$$

4 a)



b) $a \gg L$

$\Rightarrow$ like infinite plane with a strip of potential V



$\Rightarrow$ scale length above surface where $\Phi \neq 0$ is $\sim L$

$\Rightarrow E \sim \dfrac{V}{L}$

pressure $\sim \frac{1}{2}\epsilon_0 E^2$

$\sim \frac{1}{2}\epsilon_0 \dfrac{V^2}{L^2}$

$\Rightarrow$ <u>outward force</u>

$\Rightarrow$ scale length L in both z and $r-a$.

c) potential in region $\rho > a$

$\Rightarrow$ solve Laplace's Eqn with $\frac{\partial}{\partial \varphi} = 0$

$$\Phi = \int_0^\infty dk \cos kz \, K_0(k\rho) \, g_k$$

$\Rightarrow$ even around $z=0$

$\Rightarrow$ $K_0 \rightarrow 0$ as $\rho \rightarrow \infty$

$\Rightarrow$ match BC at $\rho = a$

$$\Phi(\rho = a, z) = \int_0^\infty dk \cos kz \, K_0(ka) \, g_k$$

multiply by $\cos k'z$ and integrate over z

$\Rightarrow$ note k, k' both positive

$$V \int_{-\frac{L}{2}}^{\frac{L}{2}} dz \cos k'(z) = V \frac{\sin k'z}{k'} \Big|_{-\frac{L}{2}}^{\frac{L}{2}} = 2V \frac{\sin k'\frac{L}{2}}{k'}$$

$$= \int_0^\infty dk \, g_k \cancel{\cos kz} \, K_0(ka) \underbrace{\int_{-\infty}^{\infty} dz \cos k'z \cos kz}_{I}$$

$$I = \int_{-\infty}^{\infty} dz \left(\frac{e^{ik'z} + e^{-ik'z}}{2}\right)\left(\frac{e^{ikz} + e^{-ikz}}{2}\right)$$

$$= \frac{\pi}{2}\left[2\,\cancelto{0}{\delta(k+k')} + 2\,\delta(k-k')\right] = \pi\,\delta(k-k')$$

$$2V \frac{\sin\frac{kL}{2}}{k} = \pi\, g_k K_0(ka)$$

$$g_k = \frac{2V}{\pi} \frac{\sin(kL/2)}{k K_0(ka)}$$

$$\boxed{\Phi = \frac{2V}{\pi} \int_0^\infty dk \, \frac{\sin(kL/2)}{k} \cos kz \, \frac{K_0(k\rho)}{K_0(ka)}}$$

d) Take $a \gg L$ and $\rho > a$

$k\rho \sim ka \gg 1 \Rightarrow$ expand K_0 for large argument

$$\Phi = \frac{2V}{\pi} \int_0^\infty dk \, \frac{\sin(\frac{kL}{2})}{k} \cos kz \; e^{-k(\rho-a)}$$

remaining scale length is L

Let $\frac{kL}{2} \equiv s$

$$\Phi = \frac{2V}{\pi} \int_0^\infty ds \, \frac{\sin s}{s} \cos\left(\frac{2z}{L} s\right) e^{-\frac{(\rho-a)2s}{L}}$$

For $z = 0$ ~~...~~

$$\Phi = \frac{2V}{\pi} \int_0^\infty ds \, \frac{\sin s}{s} \, e^{-(\rho-a)\frac{2s}{L}}$$

$$e' \equiv e - a$$

$$E_e = -\frac{\partial \Phi}{\partial e} = \frac{4V}{\pi L}\int_0^\infty ds\ \sin(s)\ e^{-e'\frac{2s}{L}}$$

$$= \frac{4V}{\pi L}\ \mathrm{Im}\int_0^\infty ds\ e^{is}\ e^{-e'\frac{2}{L}s}$$

$$= \frac{4V}{\pi L}\ \mathrm{Im}\ \left.\frac{e^{is}\,e^{-e'\frac{2}{L}s}}{i - e'\frac{2}{L}}\right|_0^\infty$$

$$= -\frac{4V}{\pi L}\ \mathrm{Im}\left(\frac{1}{i - \frac{2}{L}e'}\right)$$

$$= -\frac{4V}{\pi L}\ \mathrm{Im}\ \frac{-i - \frac{2}{L}e'}{1 + \frac{4}{L^2}e'^2}$$

$$\boxed{E_e = \frac{4V}{\pi L}\ \frac{1}{1 + \frac{4}{L^2}e'^2}}$$

$$= \frac{4V}{\pi L} \quad \text{for } e' = 0$$

$$\sim \frac{V}{L}$$