Physics 611 | Homework #1 | Spring '22
Dr. Drake

1. Review the notes in the file Review1.pdf through page 9. This file is in the Files directory on Canvas. This reviews the material on electrostatics covered in Physics 610. The material in these notes on basis functions in cylindrical coordinates will be covered during the first two weeks of Physics 611.

2. Consider a hollow conducting box with lengths L in the x and y directions, which has a closed bottom at $z = 0$ and which extends to ∞ in the positive z direction. The box is grounded. A charge q is placed at $x = y = L/2$ a distance d from the bottom.

   (a) Sketch the electric field lines in the $x - z$ plane at $y = L/2$ for $d \ll L$ and $d \gg L$.

   (b) What is the direction of the force on the charge q? Estimate this force when $d \ll L$. When $d \gg L$ will the force be greater or less than that which you would find if the charge were a distance d from an infinite plane conductor? Why?

   (c) Calculate the potential ϕ inside of the box.

   (d) Calculate the force acting on the charge q. Evaluate this force approximately in the limit when $d \gg L$. Is your answer in this limit consistent with your answer to the question in part (a)?
   Hint: Remember that the charge can not accelerate itself.

3. Consider a spherical conductor of radius R that is cut at $\theta = \pi/2$. The top of the conductor is maintained at a potential V and the bottom at potential $-V$. All of the following questions relate to the region $r > R$.

   (a) Sketch the electric field lines produced by the structure. What is the scaling of the electric field for $r \gg R$? Estimate the electric field around the axis of symmetry just above the conducting surface.

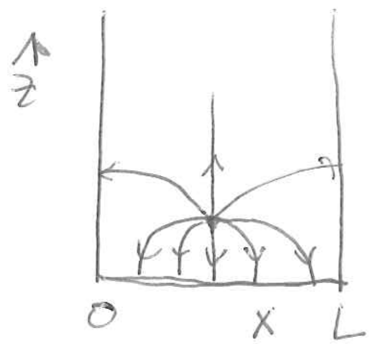
   (b) Calculate the potential ϕ in the region $r > R$.

   (c) Evaluate E at the symmetry axis just outside the conducting surface and compare with your answer in (a).

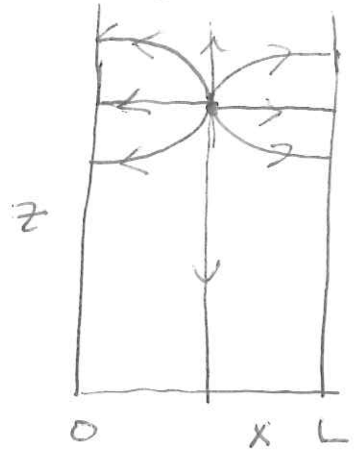
Hwk # 1 Solutions

(2) a) sketch electric field

$d << L$



$d >> L$



b) There will be induced charge on the bottom ($z=0$) of the box. This charge is negative and will produce a force downward (negative z direction). For $d >> L$ most of the induced charge will be on the side walls and most of the forces will be in the x and y directions but will balance. The net charge on the bottom will be much smaller than q so the downward force will be small compared from an infinite plane.

c) Calculate the potential inside the box

$$\varphi = \sum_{m,n} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi y}{L}\right) g_{mn}(z)$$

$$\nabla^2 \varphi = -\frac{q}{\epsilon_0}\, \delta\left(x-\frac{L}{2}\right) \delta\left(y-\frac{L}{2}\right) \delta(z-d)$$

$$\Rightarrow \quad k_{mn}^2 \equiv (m^2+n^2)\frac{\pi^2}{L^2}$$

$$\sum_{m,n} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi y}{L}\right) \left[\frac{\partial^2}{\partial z^2} g - k_{mn}^2 g\right]$$

$$= -\frac{q}{\epsilon_0}\, \delta\left(x-\frac{L}{2}\right) \delta\left(y-\frac{L}{2}\right) \delta(z-d)$$

multiply by $\sin\left(\frac{m'\pi x}{L}\right) \sin\left(\frac{n'\pi y}{L}\right)$ and integrate over x, y

$\Rightarrow$ eliminates sum over m, n

$$\frac{1}{2} L^2 \left(\frac{\partial^2}{\partial z^2} - k_{m'n'}^2\right) g(z) = -\frac{q}{\epsilon_0} \sin\left(\frac{m'\pi}{2}\right) \sin\left(\frac{n'\pi}{2}\right) \times \delta(z-d)$$

Let $m' \to m$, $n' \to n$

For $z > d$

$$\left(\frac{\partial^2}{\partial z^2} - k_{mn}^2\right) g = 0 \quad \Rightarrow \quad g = g_d\, e^{-k_{mn}(z-d)}$$

$\Rightarrow$ bounded as $z \to \infty$

For $z < d$

$$g = g_d \frac{\sinh(k_{mn} z)}{\sinh(k_{mn} d)} \quad \Rightarrow \text{continuity with } z > d$$

For $z \simeq d$ $\Rightarrow$ neglect k_{mn}^2 compared to $\frac{\partial^2}{\partial z^2}$

$$\frac{\partial^2}{\partial z^2} g = -\frac{2q}{L^2 \epsilon_0} \sin\left(\frac{m\pi}{2}\right) \sin\left(\frac{n\pi}{2}\right) \delta(z-d)$$

$$\left.\frac{\partial g}{\partial z}\right|_{d-\epsilon}^{d+\epsilon} = -\frac{2q}{L^2 \epsilon_0} \sin\left(\frac{m\pi}{2}\right) \sin\left(\frac{n\pi}{2}\right)$$

$$\left[-k_{mn} - k_{mn} \coth(k_{mn} d)\right] g = (\;)$$

$$g_d \;\cancel{g_{mn}} = +\frac{2q}{L^2 \epsilon_0 k_{mn}} \frac{\sin\left(\frac{m\pi}{2}\right) \sin\left(\frac{n\pi}{2}\right)}{1 + \coth(k_{mn} d)}$$

$\Rightarrow$ m, n odd

$\Rightarrow$ $\sin\left(\frac{m\pi}{2}\right) \Rightarrow (-1)^{m+1}$

$\sin \frac{n\pi}{2} \Rightarrow (-1)^{n+1}$

$$\varphi = \sum_{\substack{m,n\\ \text{odd}}} \frac{\frac{2q}{\epsilon_0 L^2 k_{mn}} (-1)^{m+n}}{1 + \coth(k_{mn} d)} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi y}{L}\right)$$

$$\times \quad e^{-k_{mn}(z_> - d)} \frac{\sinh(k_{mn} z_<)}{\sinh(k_{mn} d)}$$

$z_>$ = larger of z, d

$z_<$ = smaller of z, d

d) To calculate F_m need E_z produced by charges on conducting surfaces.

$\Rightarrow$ by symmetry F_m is along z.

$\Rightarrow$ need to subtract self field of q.

$\Rightarrow$ add E_z above and below q to subtract self field

$$E_z^{ext} = \frac{1}{2}\left[E_z(z=d+\epsilon) + E_z(z=d-\epsilon)\right]$$

$$E_{ext} = \sum_{\substack{m,n \\ odd}} \frac{q\,\cancel{(-1)^{(m+n)z}}}{\epsilon_0 L^2\left[1+\coth(K_{mn}d)\right]}\left[-1+\coth(K_{mn}d)\right]$$

$\Rightarrow$ note units are correct

$$E_{ext} \sim \frac{q}{\epsilon_0 L^2}$$

$\Rightarrow$ Is E_{ext} bounded ?

$\Rightarrow$ as $m,n \to \infty$

$$\coth(\,) = \frac{e^{K_{mn}d} + e^{-K_{mn}d}}{e^{K_{mn}d} - e^{-K_{mn}d}}$$

$$\simeq 1 + 2e^{-K_{mn}d}$$

For large m,n the summed terms scale like

$$\sim e^{-K_{mn}d} \Longrightarrow 0$$

$\Rightarrow$ the summed series is bounded.

Large d ? $\Rightarrow$ $K_{mn}d \gg$ for all m,n

$$E_{ext} \simeq \frac{q}{\epsilon_0 L^2}\,\frac{1}{\cancel{2}}\sum_{m,n}\cancel{2}\,e^{-K_{mn}d}$$ $\Rightarrow$ $m=n=1$ dominate

$$E_{ext} \simeq \frac{q}{\epsilon_0 L^2} e^{-\frac{\sqrt{2}\pi d}{L}}$$

$$<< \frac{q}{\epsilon_0 d^2} \quad \text{since } e^{-\frac{\sqrt{2}\pi d}{L}} << 1$$

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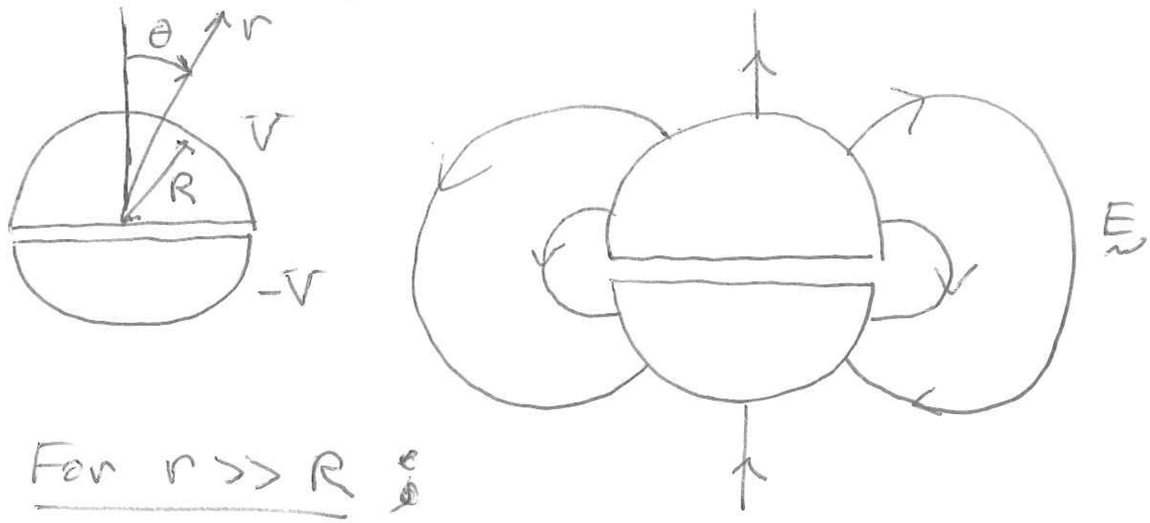
a)



<u>For $r >> R$:</u>

The upper hemisphere will have a net positive charge

$$q \sim \sigma R^2 \sim \epsilon_0 E(R) R^2$$
$$\sim \epsilon_0 \left(\frac{V}{R}\right) R^2 \sim \epsilon_0 V R$$

Lower hemisphere has opposite charge
$\Rightarrow$ acts like dipole at large r.

$$E \sim \frac{q}{\epsilon_0} \frac{1}{r^2}\left(\frac{R}{r}\right) \sim \frac{\epsilon_0 V R}{\epsilon_0} \frac{R}{r^3}$$

$$E \sim V \frac{R^2}{r^3}$$

For $r \gtrsim R$ at $\theta = 0$,

$$E \sim \frac{2V}{R\pi} \sim \frac{V}{R}$$ since potential drop is $2V$ over a distance $\sim \pi R$

b) Find the potential for $r > R$.

$$\Phi = \sum_{\ell \text{ odd}} a_\ell P_\ell(\cos\theta) \frac{R^{\ell+1}}{r^{\ell+1}}$$

since $\frac{\partial}{\partial\varphi} = 0$ and $\Phi \to 0$ as $r \to \infty$

ℓ odd because odd around $\frac{\pi}{2}$.

Calculate a_ℓ from $\Phi(r=R)$

$$\Phi(r=R) = \sum_\ell a_\ell P_\ell(\cos\theta)$$

Multiply by $P_{\ell'}(\cos\theta)$ and integrate $0,\pi$

$$\int_{-1}^{1} d\cos\theta \, P_{\ell'}(\cos\theta)\, \Phi(r=R) = \sum_\ell a_\ell \int_{-1}^{1} d\cos\theta \, P_\ell(\cos\theta) P_{\ell'}(\cos\theta)$$

$$= \sum_\ell a_\ell \delta_{\ell\ell'} \frac{2}{2\ell+1}$$

$$V \, 2 \underbrace{\int_0^1 d\cos\theta \, P_{\ell'}(\cos\theta)}_{\int_0^1 dx \, P_{\ell'}(x)} = a_{\ell'} \frac{2}{2\ell'+1}$$

$$\underbrace{\int_0^1 dx \, P_{\ell'}(x)}_{I_{\ell'}}$$

Use Rodriguez' formula $P_\ell = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell}(x^2-1)^\ell$

$$I_\ell = \frac{1}{2^\ell \ell!} \frac{d^{\ell-1}}{dx^{\ell-1}} (x^2-1)^\ell \Big|_0^1$$

zero at $x = 1$ since $\frac{d^{\ell+1}}{dx^{\ell-1}} (x-1)^\ell (x+1)^\ell \sim x-1 \Rightarrow 0$

see p. 63 and 64 of the class notes

$$I_\ell = -\frac{(-1)^{\frac{\ell+1}{2}}(\ell-1)!}{\left(\frac{\ell+1}{2}\right)!\left(\frac{\ell-1}{2}\right)!}$$

$$a_\ell = (2\ell+1)\, V\, I_\ell$$

$$\Phi = V \sum_{\ell \text{ odd.}} (2\ell+1)\, I_\ell\, P_\ell(\cos\theta) \left(\frac{R}{r}\right)^{\ell+1}$$

c) Evaluate E for $\cos\theta = 1$ and $r = R$.

$$P_\ell(1) = 1$$

$$E = V \sum_{\ell \text{ odd}} (2\ell+1)(\ell+1)\, \frac{1}{R}\, I_\ell$$

$$E \sim \frac{V}{R} \Rightarrow \text{scaling ok}$$