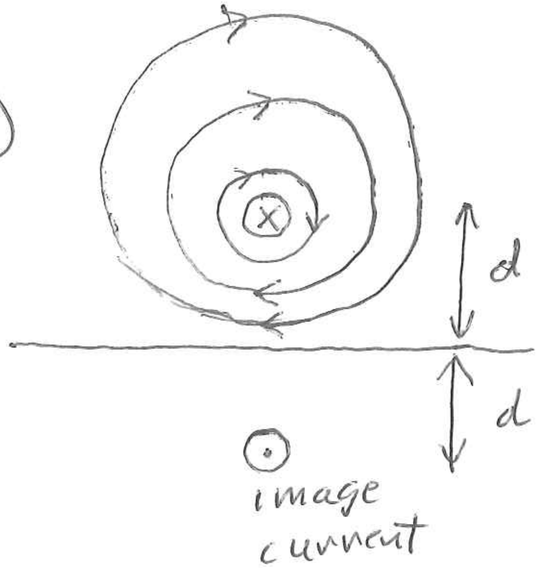
# Physics 606 Final Exam Solutions

1) a)




B at I due to image current

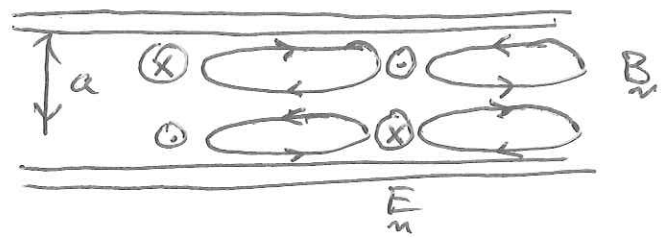
$$B = \frac{\mu_0 I}{2\pi(2d)}$$

force/length $= IB = \lambda g$

$$\boxed{\frac{\mu_0 I^2}{4\pi d} = \lambda g}$$

TE mode $\Rightarrow E_z = 0,\ B_z \neq 0$

b)



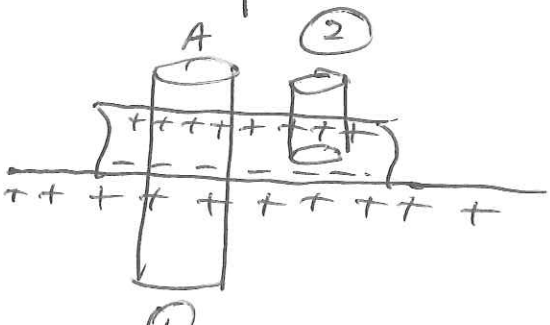

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right) B_z = 0$$

$$\nabla^2 = \nabla_t^2 - k_z^2 \sim -\frac{1}{a^2} - k_z^2 = -\frac{\omega^2}{c^2}$$

$$\Rightarrow k_z^2 = \frac{\omega^2}{c^2} - \frac{1}{a^2} > 0$$

$$\Rightarrow \omega^2 > \frac{c^2}{a^2}$$

c)


dielectric is polarized

$E_0$ outside dielectric use Gaussian surface ①

$2AE_0 = \frac{\sigma}{\epsilon_0}A \qquad E_0 = \frac{\sigma}{2\epsilon_0}$

Use Gaussian surface ② to calculate $E_i$ inside dielectric. Since no free charge

$$D_0 = D_i = \epsilon_0 E_0 = \epsilon_i E_i$$

$$\Rightarrow E_i = \frac{\epsilon_0}{\epsilon_i} E_0 < E_0$$

But $\nabla \cdot \underset{\sim}{E} = \frac{\rho_{tot}}{\epsilon_0}$  For ② $\rho_{tot} = \rho_p$ = pol charge

$$E_0 - E_i = \frac{1}{\epsilon_0}\sigma_{pol} = \frac{\sigma}{2\epsilon_0}\left(1 - \frac{\epsilon_0}{\epsilon_i}\right)$$

$$\sigma_{pol} = \frac{\sigma}{2}\left(1 - \frac{\epsilon_0}{\epsilon_i}\right) \Rightarrow \text{top of dielectric}$$

$$\Rightarrow -\sigma_{pol} \Rightarrow \text{bottom of dielectric}$$

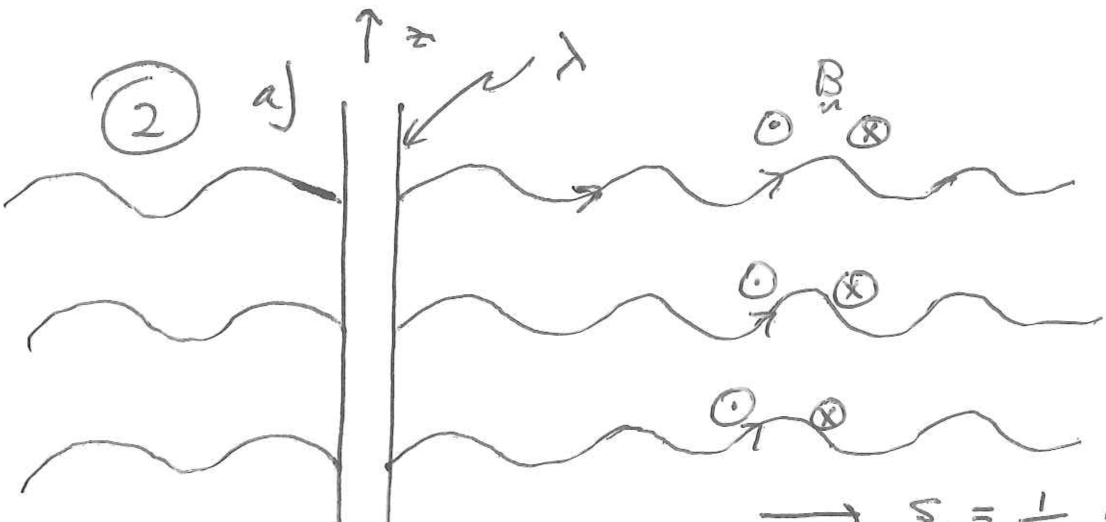
Calculate force on $\sigma_{pol}$ and $-\sigma_{pol}$ due to $\sigma$.

~~F ⇒~~

$\Rightarrow$ E due to $\sigma$ is uniform so forces on $\sigma_{pol}$ and $-\sigma_{pol}$ cancel

$\Rightarrow$ no net force

2) a)




$$\longrightarrow \quad \underset{\sim}{S} = \frac{1}{\mu_0}\,\underset{\sim}{E} \times \underset{\sim}{B}$$

$\underset{\sim}{E}$ is along $\hat{z}$

$\underset{\sim}{B}$ is in $\varphi$ direction

$\vec{S}$ is radial

b) $\Rightarrow$ electromagnetic wave

$$E_z = -c\,B_\varphi$$

total power from radiation field is independent of $\rho$

$$\Rightarrow 2\pi\rho\, S \sim \rho E^2 \sim \text{const}$$

$$E, B \sim \frac{1}{\rho^{1/2}}$$

c) $$\nabla \times \underset{\sim}{B} = \mu_0 \underset{\sim}{J} + \epsilon_0 \frac{\partial}{\partial t} \underset{\sim}{E}\, \mu_0$$

$$\underset{\sim}{B} = \nabla \times \underset{\sim}{A} \qquad \underset{\sim}{J} = J_z \hat{z}$$

$$\underset{\sim}{E} = -\frac{\partial}{\partial t} \underset{\sim}{A} - \nabla \varphi \implies E_z = -\frac{\partial}{\partial t} A_z$$

$$\left[\nabla \times (\nabla \times \underset{\sim}{A})\right]_z = \left[\nabla\, \nabla \cdot \underset{\sim}{A} - \nabla^2 \underset{\sim}{A}\right]_z$$

$$= \frac{\partial}{\partial z} \frac{\partial}{\partial z} A_z - \nabla^2 A_z$$

$$\frac{\partial}{\partial z} = 0$$

$$-\nabla^2 A_z = \mu_0 J_z - \cancel{\mu_0 \epsilon_0}\, \ddot{A}_z \frac{1}{c^2}$$

$$\nabla^2 A_z + \frac{\omega^2}{c^2} A_z = \mu_0 J_z$$

$$J_z = \rho V_z = -i\omega d\, \cancel{\rho}\, \lambda\, \delta(\rho - a) \frac{1}{2\pi a}$$

$$\frac{1}{\ell} \int d\underset{\sim}{x}\, \rho = \lambda = \frac{\ell\, 2\pi}{\ell} \int_0^\infty d\rho\, \rho\, \rho$$

$$\implies \rho = \frac{\lambda}{2\pi a} \delta(\rho - a)$$

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} A_z + \frac{\omega^2}{c^2} A_z = \mu_0 \left(-\frac{i\omega d \lambda}{2\pi a} \delta(\rho - a)\right)$$

$$\boxed{A_z'' + \frac{1}{\rho} A_z' + \frac{\omega^2}{c^2} A_z = S_0\, \delta(\rho - a)}$$

$\implies$ Bessel Eqn with $\nu = 0$, $k = \frac{\omega}{c}$

$$S_0 = -\frac{i\omega d \lambda \mu_0}{2\pi a}$$

d) $\rho > a$

$$A_z'' + \frac{1}{\rho} A_z' + \frac{\omega^2}{c^2} A_z = 0$$

B.C. $\Rightarrow$ out going wave

$$A_z \sim H_0^{(1)} \sim \frac{e^{ik\rho}}{\rho^{1/2}} e^{-i\omega t} \quad \Rightarrow \text{outgoing}$$

$$A_z = C^+ H_0^{(1)}\left(\frac{\omega}{c}\rho\right)$$

$\rho < a$

$\Rightarrow$ bounded at $\rho = 0$

$\Rightarrow$ $J_0(k\rho)$

$$A_z = C^- J_0\left(\frac{\omega}{c}\rho\right)$$

Integration eqn for $A_z$ across $\rho = a$

$$\left.\frac{\partial A_z}{\partial \rho}\right|_{a-\epsilon}^{a+\epsilon} = S_0'$$

$$\left. A_z \right|_{a-\epsilon}^{a+\epsilon} = 0 \quad \Rightarrow \quad C^+ = C^0 J_0\left(\frac{\omega}{c}a\right)$$

$$C^- = H_0^{(1)}\left(\frac{\omega}{c}a\right) C^0$$

$$C^0 \frac{\omega}{c}\left[ H_0^{(1)\prime} J_0\left(\frac{\omega}{c}a\right) - H_0^{(1)}\left(\frac{\omega}{c}a\right) J_0'\left(\frac{\omega}{c}a\right)\right] = S_0'$$

$\underline{\rho > a}$ $\quad A_z = c^0 J_0\left(\frac{\omega}{c}a\right) H_0^{(1)}\left(\frac{\omega}{c}\rho\right)$

$\underline{\rho < a}$ $\quad A_z = c^0 J_0\left(\frac{\omega}{c}\rho\right) H_0^{(1)}\left(\frac{\omega}{c}a\right)$

$$c^0 = -\frac{i\,d\,\lambda\,\mu_0\,c}{2\pi a\,\omega}\Big[\underbrace{H_0^{(1)\prime} J_0 - H_0^{(1)} J_0'}_{i(N_0' J_0 - N_0 J_0')}\Big]^{-1}_{\rho=a}$$

$$i\,\frac{2}{\pi x} = i\,\frac{2c}{\pi\omega a}$$

$$\boxed{c^0 = -\frac{\lambda d \mu_0}{4}\,\omega}$$

e) $\underset{\sim}{B} = \nabla \times \underset{\sim}{A}$

$$B_\varphi = -\frac{\partial}{\partial\rho} A_z$$

$$A_z \simeq c^0 J_0\left(\frac{\omega}{c}a\right)\sqrt{\frac{2}{\pi\frac{\omega}{c}\rho}}\; e^{i\frac{\omega}{c}\rho - i\frac{\pi}{4}}$$

$$B_\varphi = -i\frac{\omega}{c}\,c^0 J_0\left(\frac{\omega}{c}a\right)\sqrt{\frac{2}{\pi\frac{\omega}{c}\rho}}\; e^{i\frac{\omega}{c}\rho - i\frac{\pi}{4}}$$

$$E_z = -c B_\varphi$$

f) $$S_\rho = -\frac{1}{2}\frac{1}{\mu_0}\,\mathrm{Re}\left(E_z B_\varphi^*\right)$$

$$= \frac{1}{2\mu_0}c\,\frac{\omega^2}{c^2}|c^0|^2 J_0^2\left(\frac{\omega}{c}a\right)\frac{2}{\pi\frac{\omega}{c}\rho}$$

total power

$$P \sim S_\rho\, 2\pi\rho \Rightarrow \text{indep of } \rho$$

3

a) $\underset{\sim}{J}'(x') = K\,\delta(x')\,\hat{y}$

$\nabla \times \underset{\sim}{B}' = \mu_0 K\,\delta(x')\,\hat{y}$

$-\frac{\partial}{\partial x} B_z' = \mu_0 K\,\delta(x')$

$\boxed{B_z' = -\frac{\mu_0 K}{2} H(x')}$

define $H = 1 \quad x' > 0$
$= -1 \quad x' < 0$

$\rho' = 0$ and $\frac{\partial}{\partial t'} = 0 \Rightarrow \boxed{\underset{\sim}{E}' = 0}$

b) Lab frame $S' \Rightarrow (c\rho, J)$ are a 4-vector

$c\rho = \gamma(c\rho' + \beta J_x') = 0$

$J_y = J_y'$

$\boxed{J_y = K\,\delta(\gamma(x - \beta ct))}$

$x' = \gamma(x - \beta ct)$

$\boxed{\rho = 0}$

c) $\frac{1}{c} E_y = \gamma \left[ \cancel{\frac{1}{c} E_y'}^{\,0} - \underset{\sim}{\beta} \times \underset{\sim}{B}'_\perp \right]_y$

$= \gamma \beta B_z'$

$\boxed{E_y = -\frac{c\gamma\beta\mu_0 K}{2} H(\gamma(x - \beta ct))}$

$$B_z = \gamma\left(B_z' + \cancel{\underset{\sim}{\beta} \times \frac{1}{c} \underset{\sim}{E}_\perp'}\right) \quad 0$$

$$\boxed{B_z = -\gamma \frac{\mu_0 K}{2} H(\gamma(x - \beta c t))} \qquad \boxed{E_y = v B_z}$$

d) $\nabla \times \underset{\sim}{B} = \mu_0 \underset{\sim}{J} + \frac{1}{c^2} \frac{\partial}{\partial t} \underset{\sim}{E}$

$$-\frac{\partial}{\partial x} B_z = \mu_0 J_y + \frac{1}{c^2} \frac{\partial}{\partial t} E_y$$

$$\nabla \times \underset{\sim}{E} + \frac{\partial}{\partial t} \underset{\sim}{B} = 0$$

$$\frac{\partial}{\partial x} E_y + \frac{\partial}{\partial t} B_z = 0$$

$$J_y = J_x(x - vt)$$

$$\Longrightarrow \frac{\partial}{\partial t} = -v \frac{\partial}{\partial x}$$

$$\frac{\partial}{\partial x} E_y - v \frac{\partial}{\partial x} B_z = 0$$

$$\boxed{E_y = v B_z} \qquad \boxed{B_z = -\frac{\mu_0 \gamma K}{2} \otimes H(\gamma(x - vt))}$$

$$-\frac{\partial}{\partial x} B_z = \mu_0 J_y + \frac{1}{c^2}\left(-v \frac{\partial}{\partial x}\right) v B_z$$

$$\frac{\partial}{\partial x} B_z = -\mu_0 \gamma^2 J_y$$

$$\Delta B_z = -\mu_0 \gamma^2 K \int dx\, \delta(\gamma(x - vt))$$