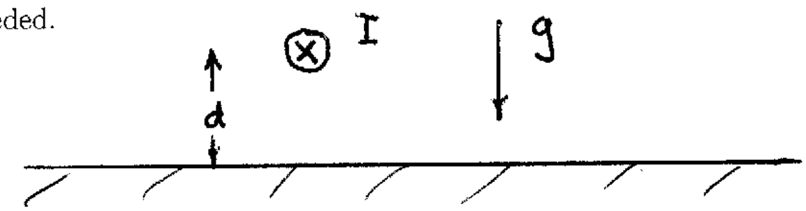
Physics 606 Final Exam Spring '21

Dr. Drake

1. (60 points) The following are short answer questions which should not require extensive calculations.

   (a) An infinite wire carrying a current I and mass-per-unit-length λ is brought from infinity and comes to an equilibrium a distance d above a perfect conductor in a gravitational field g. Take the current I to be constant. Sketch the magnetic field in the plane perpendicular to the wire in the region above the conducting surface. Calculate the magnetic field at the location of the wire due to the image current. In the equilibrium state calculate a relation between I, g and d and whatever other universal constants are needed.



   (b) A TE mode of frequency ω propagates down a hollow cylindrical ideal conductor of radius a. Sketch the electric and magnetic fields for the lowest order mode. Estimate the minimum frequency at which energy will propagate down the system. Explain your reasoning.
   Hint: don't work out the full solutions for the TE mode. An estimate for the lowest frequency is sufficient.

   (c) Consider an infinite sheet of charge per unit area σ. Assume that $\sigma > 0$. A dielectric of dielectric constant ϵ and uniform thickness d is placed on top of the charged sheet. Sketch the induced charge that is generated due to the polarization of the dielectric. Calculate the electric field E_0 in the region above the dielectric. Calculate E inside the dielectric. Calculate the induced charge on the dielectric? What is the net force between the dielectric and the charged sheet?

2. (70 points) Consider an infinite dielectric rod with charge per unit length λ and radius "a" with $\mu = \mu_0$. The rod is aligned with the z axis. The rod oscillates along the z direction with an amplitude d and frequency ω. The displacement

Δz is given by $\Delta z = de^{-i\omega t}$. Assume that the charge lies on the surface of the rod.

(a) Sketch the electric field in the $z - \rho$ plane. What is the direction of the magnetic field? The Poynting flux?

(b) In the region $\omega\rho/c \gg 1$ what are the relative magnitudes of the oscillatory components of **E** and **B**? How do they fall off with distance ρ in the same region?

(c) Derive an equation for the vector potential A_z produced by the oscillating current J_z.

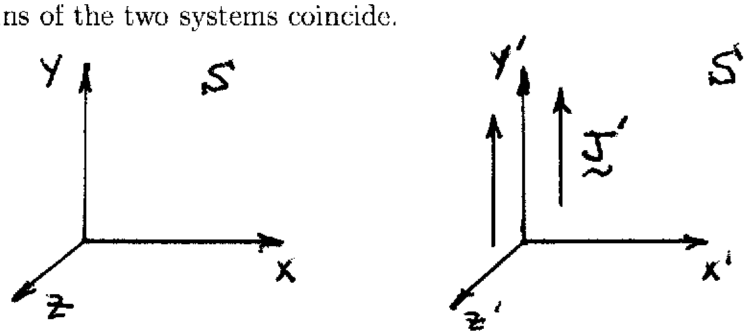
(d) Solve this equation for A_z in the region $\rho > a$. What is the boundary condition on the wave in this region? Solve for A_z in the region $\rho < a$. Match the solutions at $\rho = a$. Use the Wronskian below to simplify your answer.

$$J_\nu N_\nu' - N_\nu J_\nu' = \frac{2}{\pi x}$$

(e) Calculate **E** and **B** in the region $\omega\rho/c \gg 1$.

(f) Calculate the time-average Poynting flux in this same region. Show that the total radiated power is independent of ρ.

3. (70 points) An infinite current sheet of current density $\mathbf{J}'(x') = K\delta(x')\hat{y}$ is at rest in the S' system. The S' system is moving along the positive x axis with a uniform velocity v with respect to the laboratory system S. At $t = t' = 0$ the origins of the two systems coincide.



(a) Calculate $\mathbf{E}'$ and $\mathbf{B}'$ in the rest frame of the current sheet. Express your answer in terms of $G(x')$, where $G = 1$ for $x' > 0$ and $G(x') = -1$ for $x' < 0$.

(b) Calculate the charge $\rho(x,t)$ and current $\mathbf{J}(x,t)$ densities as seen in the lab frame S. Express your answer in terms of the laboratory space/time coordinates.

(c) Calculate $\mathbf{E}$ and $\mathbf{B}$ as seen by the observer in the lab frame by transforming $\mathbf{E}'$ and $\mathbf{B}'$ directly. Again express your answer in the lab space/time coordinates.

(d) Calculate $\mathbf{E}$ and $\mathbf{B}$ directly in the lab frame by solving Maxwell's equations with ρ and $\mathbf{J}$ as sources.

Hint: in the lab frame $\partial/\partial t$ and $\partial/\partial x$ are related. Use this to simplify the equations.

Equation Sheet Physics 606

Maxwell's Equations

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial}{\partial t}\mathbf{D} \qquad \mathbf{D} = \epsilon \mathbf{E} = \epsilon_0 \mathbf{E} + \mathbf{P}$$

$$\nabla \times \mathbf{E} + \frac{\partial}{\partial t}\mathbf{B} = 0 \qquad \mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) = \mu \mathbf{H}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \mathbf{B} = \nabla \times \mathbf{A}, \quad \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{A} - \nabla \varphi$$

$$\nabla \cdot \mathbf{D} = \rho \qquad \epsilon = \epsilon_0(1 + \chi_e), \quad \rho_p = -\nabla \cdot \mathbf{P}$$

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$$

Math eqns

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\sin\theta \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \varphi^2}$$

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}$$

$$\nabla^2 G(\mathbf{x}, \mathbf{x}') = -4\pi\,\delta(\mathbf{x} - \mathbf{x}'), \qquad \delta(\mathbf{x}) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} d\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}}$$

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{|\mathbf{x} - \mathbf{x}'|} \qquad \int_{-\infty}^{\infty} dx\,\delta[f(x)] = \frac{1}{|f'(0)|}$$

Bessel Eqn: $\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} + \left(1 - \frac{\nu^2}{x^2}\right) = 0$

$$J_\nu(x) \sim x^\nu \text{ (small } x) \sim \frac{1}{x^{1/2}}\cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \text{ (large}$$

$$N_\nu(x) \sim x^{-\nu} \text{ (small } x) \sim \frac{1}{x^{1/2}}\sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \text{ (large}$$

$$\sim \ln x \ (\nu = 0)$$

$$H_\nu^{(1)}(x) = J_\nu + iN_\nu, \qquad H_\nu^{(2)}(x) = J_\nu - iN_\nu$$

Mod Bessel Eqn: $\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} - \left(1 + \frac{\nu^2}{x^2}\right) = 0$

$$I_\nu(x) \sim x^\nu \text{ (small } x) \sim \frac{1}{x^{1/2}} e^{x} \text{ (large } x)$$

$$K_\nu(x) \sim x^{-\nu} \text{ (small } x) \sim \frac{1}{x^{1/2}} e^{-x} \text{ (large } x)$$

$$\sim \ln x \ (\nu = 0)$$

$$\int_0^a d\rho\, \rho\, J_\nu^2\left(x_{\nu n}\frac{\rho}{a}\right) = \frac{a^2}{2} J_{\nu+1}^2(x_{\nu n})$$

Legendre Eqn: $\frac{d}{dx}(1 - x^2)\frac{d}{dx} + \left[\ell(\ell+1) - \frac{m^2}{1 - x^2}\right] = 0$

$\int d\Omega \, |Y_{\ell m}|^2 = 1$, $Y_{\ell m} = \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}} \, P_\ell^m(\cos\theta) \, e^{im\varphi}$

Waves

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}$$

$$\Box^2 G = \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G = -4\pi \delta(\mathbf{x}-\mathbf{x}') \otimes \delta(t-t')$$

$$G = \frac{1}{|\mathbf{x}-\mathbf{x}'|} \delta\left(t' - \left(t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)\right)$$

$$\mathbf{P} = \frac{1}{c^2} \mathbf{E} \times \mathbf{H}$$

wave eqn: $\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\begin{pmatrix}\mathbf{E} \\ \mathbf{B}\end{pmatrix} = 0$

Electrostatics

charge: $\varphi(\mathbf{x}) = \frac{q}{4\pi\epsilon_0 |\mathbf{x}|}$

$\mathbf{F} = q\mathbf{E}$ $\quad u_E = \frac{1}{2}\epsilon E^2$

$\mathbf{E} = -\nabla\varphi$

dipole: $\varphi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p}\cdot\mathbf{x}}{|\mathbf{x}|^3}$

$\mathbf{p} = \int d\mathbf{x} \, \rho(\mathbf{x}) \, \mathbf{x}$

surface charge: $E = \frac{\sigma}{2\epsilon_0}$

$u_q = q\varphi$, $u_p = -\mathbf{p}\cdot\mathbf{E}$

magnetostatics

$d\mathbf{B} = \frac{\mu_0}{4\pi} I \frac{d\mathbf{l} \times \mathbf{x}}{|\mathbf{x}|^3}$

wire: $B = \frac{\mu_0 I}{2\pi \rho}$

$\mathbf{B} = \nabla \times \mathbf{A}$

$d\mathbf{F} = I \, d\mathbf{l} \times \mathbf{B}$

$\mathbf{A} = \frac{\mu_0}{4\pi} \frac{\mathbf{m}\times\mathbf{x}}{|\mathbf{x}|^3}$ mag. dipole

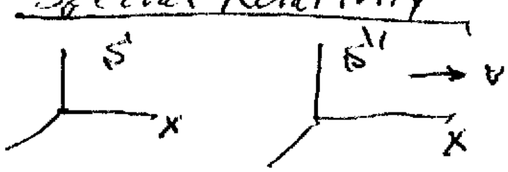
$U_m = -\mathbf{m}\cdot\mathbf{B}$

$\mathbf{m} = \frac{I}{2} \oint d\mathbf{l} \times \mathbf{x}$

$u_B = \frac{1}{2\mu_0} B^2$

$m = IA$

Special Relativity



$\beta = v/c \quad \gamma = 1/(1-\beta^2)^{\frac{1}{2}}$

$$\begin{pmatrix} x^{0\prime} \\ x^{1\prime} \\ x^{2\prime} \\ x^{3\prime} \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

4-vectors: $(ct, \mathbf{x})$, $\left(\frac{\mathcal{E}}{c}, \mathbf{p}\right)$, $\left(\frac{\omega}{c}, \mathbf{k}\right)$, $(c\rho, \mathbf{J})$

$\left(\frac{\varphi}{c^2}, \mathbf{A}\right)$ $\quad \mathcal{E} = \gamma m c^2$, $\mathbf{p} = \gamma m \mathbf{u}$

velocity: $u_x = \frac{u_x' + v}{1 + \frac{v u_x'}{c^2}}$, $\mathbf{u}_\perp = \frac{\mathbf{u}_\perp}{\gamma_v\left(1 + \frac{v u_x'}{c^2}\right)}$

fields: $E_\parallel' = E_\parallel$, $B_\parallel' = B_\parallel$

$\frac{1}{c}\mathbf{E}_\perp' = \gamma\left(\frac{1}{c}\mathbf{E}_\perp + \boldsymbol{\beta} \times \mathbf{B}_\perp\right)$, $\mathbf{B}_\perp' = \gamma\left(\mathbf{B}_\perp - \boldsymbol{\beta} \times \mathbf{E}_\perp \frac{1}{c}\right)$