Physics 606 Midterm #2 Spring '19

Dr. Drake

1. (50 points) A waveguide consists of a vacuum enclosed by a cylindrical conductor of radius "a" and infinite extent along z. Assume that the conductivity of the metal wall is infinite. The guide is excited with an antenna of frequency ω. The following questions relate to the lowest order TM ($B_z = 0$) mode of the guide.

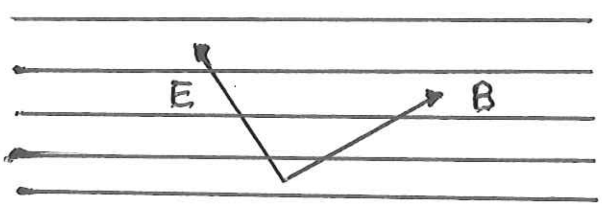
   (a) Sketch the electric and magnetic fields for this mode. What are the nonzero components of $\mathbf{E}$ and $\mathbf{B}$?

   (b) Starting with Maxwell's equations derive an equation for E_z. What boundary conditions on E_z must be applied at the conducting surface?

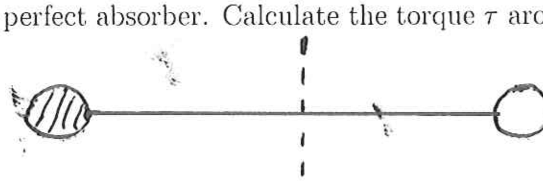
   (c) Solve the equation derived in (b) and calculate the velocity at which energy propagates down the guide. What is the lowest frequency for which energy propagates down the guide?

2. (50 points) The following are short answer questions which do not require extensive calculations.

   (a) An electromagnetic wave is incident on a grid of fine, very high conductivity wires as shown. The wavelength of the light is much greater than the spacing of the wires. What happens to the wave?



   (b) An electromagnetic wave of frequency ω and amplitude E is normally incident on two circular plates of radius "a" connected by a rigid bar of length $L \gg a$ as shown. One of the plates is a perfect conductor and the other is a perfect absorber. Calculate the torque τ around the center axis.



   (c) A cylindrical rod of radius a and length L and permeability μ is placed in an initially uniform magnetic field $\mathbf{B}_0$ that is pointed in the z direction. The axis of the rod is in the direction of the initial field. Assume that $L \gg a$.

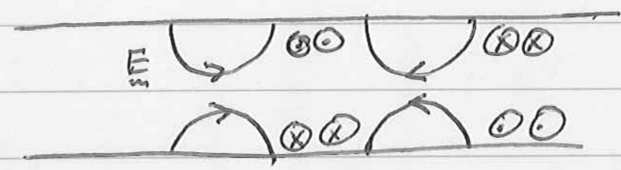
i. Under the assumption that the end effects of the finite length rod can be neglected, calculate the magnetic field $\mathbf{B}$ both inside and outside of the rod (far from the ends of the rod).
   Hint: What is the direction of $\mathbf{H}$ and $\mathbf{B}$ in this region?

ii. The rod is now cut transverse to its axis (it is now two rods with their axes aligned) and the two pieces are separated by a small distance $d \ll z$. Calculate the force between the two pieces. Is it attractive or repulsive? What is the force in the limit $\mu = \mu_0$?

Phys. 606 Solutions Midterm #2 Spring '19

1

a) TM mode lowest order $\Rightarrow \frac{\partial}{\partial \varphi} = 0$



$E_z \neq 0 \quad E_\rho \neq 0 \quad B_\varphi \neq 0 \qquad \underset{\sim}{E}, \underset{\sim}{B} \sim e^{-i\omega t} e^{ikz}$

b) $\nabla \times B = -i\frac{\omega}{c^2}\underset{\sim}{E} \qquad \nabla \cdot \underset{\sim}{E} = 0$

$\nabla \times \left(\nabla \times \underset{\sim}{E} - i\omega\, \underset{\sim}{B} = 0 \right)$

$\cancel{-}\nabla^2 \underset{\sim}{E} \cancel{+}\cancel{i}\frac{\omega}{c}\left(\cancel{+}\cancel{i}\frac{\omega}{c}\underset{\sim}{E}\right) = 0$

$$\boxed{\frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial E_z}{\partial \rho} + \left(\frac{\omega^2}{c^2} - k^2\right)E_z = 0} \qquad \boxed{\begin{array}{c} E_z = 0 \text{ at} \\ \rho = a \end{array}}$$

c) $E_z \sim J_0(\gamma\rho) \qquad \gamma^2 = \frac{\omega^2}{c^2} - k^2$

$\gamma a = x_{01} \Leftarrow J_0(\gamma a) = 0$

$\frac{\omega^2}{c^2} - k^2 = \frac{x_{01}^2}{a^2} \Longrightarrow$ gives k

Calculate group velocity

$$\frac{\cancel{\hbar}\,\omega v_g}{c^2} = \cancel{\hbar} K$$

$$v_g = \frac{c^2}{\omega}\left(\frac{\omega^2}{c^2} - \frac{x_{01}^2}{a^2}\right)^{\frac{1}{2}}$$

$$\boxed{\frac{v_g}{c} = \left(1 - \frac{x_{01}^2 c^2}{a^2 \omega^2}\right)^{\frac{1}{2}}}$$

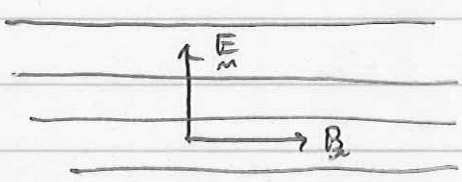
lowest frequency

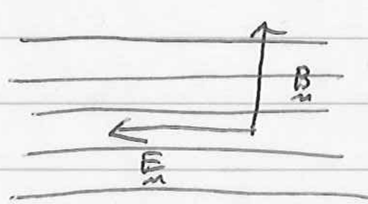
$$\boxed{\omega = \frac{x_{01}}{a} c}$$

(2)

a)

10 Separate into two polarizations with $\underset{\sim}{E}$ along and across the wires





This wave is all transmitted since no currents flow transverse to the wires

This wave is reflected since large currents flow in the wires. Acts like a conducting surface

(b) Calculate momentum ~~[scribble]~~ deposited on the disks.

momentum density of wave

10 $$P = \frac{1}{c^2}\frac{EB}{\mu_0}\frac{\epsilon_0}{\epsilon_0} \qquad Bc = E$$

$$\bar{P} = |E|^2 \frac{\epsilon_0}{2c}$$

absorber $\frac{\text{momentum}}{\text{time}} = \cancel{\qquad} = \text{force} = \bar{P} c \pi a^2$

ideal conductor reflects wave

reflector: reflected wave carries same momentum as incident so ~~the~~ reflector receives twice the momentum

$$\cancel{\boxed{\tau \simeq \frac{|E|^2 a^2}{8\pi}\,\frac{a}{2}}} \qquad \tau = \bar{P} c \pi a^2 \frac{a}{2}$$

$$= \frac{\epsilon_0 |E|^2 \pi a^3}{4}$$

~~(c)~~

$$\cancel{J = 0 \Rightarrow \nabla \times H = 0}$$

$$\cancel{H = -\nabla \varphi_m}$$

$$\cancel{\nabla \cdot B = 0 =}$$

c) Neglect end effects $\frac{\partial}{\partial z} \sim 0$

$$\underset{\sim}{B} = B_z(\rho)\,\hat{z} \qquad B_z = \mu H_z$$

$$\nabla \times \underset{\sim}{H} = 0$$

$$\nabla \times \frac{B_z(\rho)}{\mu(\rho)}\,\hat{z} = 0$$

15

$$\nabla\left(\frac{B_z}{\mu}\right) \times \hat{z} = 0$$

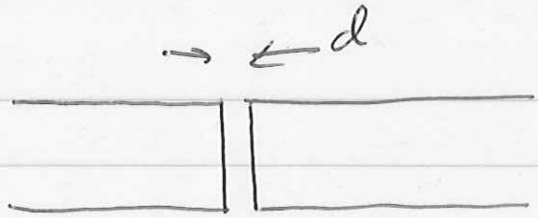
$$\cancel{B_z} \quad \frac{B_z}{\mu} = \text{const} = B_0 \frac{1}{\mu_0}$$

$$\boxed{B_z = \frac{\mu(\rho)}{\mu_0} B_0}$$

<u>inside</u>

$$B_z = \frac{\mu}{\mu_0} B_0$$

outside $B_z = B_0$



$\nabla \cdot \underset{\sim}{B} = 0 \Rightarrow B_z$ is a constant across the gap

$$= \frac{\mu}{\mu_0} B_0$$

~~Enrg~~ Energy in gap region

$$W = \frac{B_z^2}{\cancel{8\pi}} d \pi a^2 \frac{1}{2\mu_0}$$

$$= \frac{\mu^2 B_0^2}{\cancel{8\pi}} \frac{d \pi a^2}{2 \mu_0^3}$$

ΔW = change from background

$$\Delta W = \left(\frac{\mu^2}{\mu_0^2} - 1\right) \frac{B_0^2}{\cancel{8\pi}} \frac{d \pi a^2}{2\mu_0}$$

$$F = \frac{\partial W}{\partial d}$$

$$\boxed{F = \left(\frac{\mu^2}{\mu_0^2} - 1\right) \frac{B_0^2}{\cancel{8\pi}} \frac{\pi a^2}{2\mu_0}}$$

attractive since energy is smaller for smaller d. (for $\frac{\mu}{\mu_0} > 1$)

$$F = 0 \text{ for } \mu = \mu_0$$