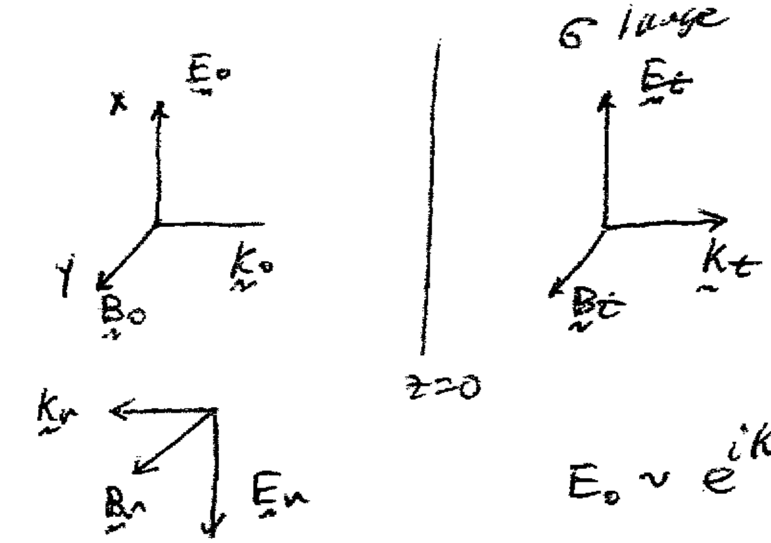
Phys. 606 Exam 2 Solutions

1

a)



$$E_0 \sim e^{ik_0 z}\, e^{-i\omega t}$$

b) calculate dispersion relation for conductor

$$\nabla\times\underset{\sim}{B} = \mu_0 \underset{\sim}{J} + \mu_0\epsilon_0 \frac{\partial}{\partial t}\underset{\sim}{E} \simeq \mu_0\sigma\,\underset{\sim}{E}$$

$$\nabla\times\underset{\sim}{E} + \frac{\partial \underset{\sim}{B}}{\partial t} = 0$$

$0 \qquad \frac{\partial}{\partial x} = 0$

$$\nabla\times\dot{\underset{\sim}{B}} = -\nabla\times(\nabla\times\underset{\sim}{E}) = -\nabla\nabla\cdot\underset{\sim}{E} + \nabla^2\underset{\sim}{E} = \mu_0\sigma\dot{\underset{\sim}{E}}$$

$$\cancel{k^2 = i\mu_0\sigma\omega}$$

$$\boxed{+k^2 = \mu_0\sigma(+i\omega)}$$

$$k = \sqrt{\frac{\mu_0\sigma\omega}{2}}\,(1+i) \equiv \frac{1}{\delta}(1+i)$$

<u>vacuum</u>

$$\nabla\times\dot{B} = \nabla^2\underset{\sim}{E} = \mu_0\epsilon_0\ddot{E}$$

$$\boxed{k^2 = \mu_0\epsilon_0\omega^2}$$

c) Since $\mu = \mu_0$ on both sides

$$\hat{n} \times \underset{\sim}{B} \Big|_{-}^{+} = 0$$

$$\hat{n} \times \underset{\sim}{E} \Big|_{-}^{+} = 0$$

d)

continuity of E_x

① $E_0 - E_r = E_t$

continuity of B_y

② $B_0 + B_r = B_t$

relate B_y to E_y

in vacuum: $\nabla \times \underset{\sim}{E} + \frac{\partial B}{\partial t} = 0$

$$ikE_x - i\omega B_y = 0$$

$$E_x = cB_y$$

in conductor:

$$i\frac{1}{\delta}(1+i)E_x = \cancel{i}\omega B_y$$

from ① and ②

$$\cancel{k}B_0 - \cancel{k}B_r = \frac{\omega B_t \delta}{(1+i)c} = \frac{k_0 \delta B_t}{1+i}$$

$$B_0 + B_r = B_t$$

$$2B_0 = B_t\left(1 + \frac{k\delta}{1+i}\right)$$

$$B_t = \frac{2B_0}{1+\frac{k\delta}{1+i}} \simeq \boxed{2B_0}$$

$$\cancel{B}\ E_t = \frac{\omega\, 2B_0\, \delta\, c}{(1+i)\, c}$$

$$\boxed{E_t = \frac{2k_0\delta}{1+i}\, cB_0}$$

$$\boxed{B_r = B_t - B_0 \simeq B_0}$$

$$\boxed{E_r \simeq cB_0}$$

2) Poynting flux

conductor:

$$S_z^t = \mathrm{Re}\, \frac{E_x B_y^*}{2\mu_0} = \mathrm{Re}\, \frac{\omega\delta}{1+i}\, \frac{|B_t|^2}{2\mu_0}\, \frac{(1-i)}{1-i}$$

$$= \frac{\omega\delta}{\mu_0 c} B_0^2\, c = k_0\delta\, \frac{B_0^2}{\mu_0} c$$

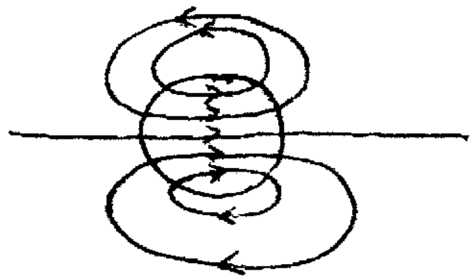
vacuum:

$$S_z^0 = \frac{1}{2\mu_0} c B_0^2$$

$$\boxed{\frac{S_z^t}{S_z^0} = 2k_0\delta}$$ fraction transmitted

$\Rightarrow$ dissipated by Joule heating

2 a) Sketch $\underset{\sim}{B}$



b) Since $\nabla \times \underset{\sim}{H} = 0$

$$\Rightarrow \underset{\sim}{H} = -\nabla \varphi_m$$

$$\underset{\sim}{B} = \mu_0 (\underset{\sim}{H} + \underset{\sim}{M}) \Rightarrow \underset{\sim}{M} \text{ specified}$$

$$\nabla \cdot \underset{\sim}{B} = 0 \quad \Rightarrow \quad \nabla^2 \varphi_m = \nabla \cdot \underset{\sim}{M}$$

$$\nabla \cdot \underset{\sim}{M} = \frac{\partial}{\partial x} M_0 H(a-\rho)$$

$$= -\frac{\partial \rho}{\partial x} M_0 \delta(a-\rho)$$

$$\rho^2 = x^2 + y^2$$

$$\cancel{2}\rho \frac{\partial \rho}{\partial x} = \cancel{2} x \quad \Rightarrow \quad \frac{\partial \rho}{\partial x} = \cos\varphi$$

$$\nabla^2 \varphi_m = -M_0 \cos\varphi \, \delta(a-\rho)$$

where $\nabla^2 = \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \varphi^2}$

Basis functions

$$\sim \sin(n\varphi), \cos(n\varphi), \rho^{\pm n}$$

$$n = 1, 2 \ldots$$

Since $\nabla \cdot \underset{\sim}{M}$ has even symmetry in φ,

φ_m is also even in φ

For $\rho > a$,

$$\varphi_m^> = \sum_n b_>^n \left(\frac{a}{\rho}\right)^n \cos n\varphi$$

For $\rho < a$

$$\varphi_m^< = \sum_n b_<^n \left(\frac{\rho}{a}\right)^n \cos n\varphi$$

Only $n=1$ yields non-zero $b_{\gtrless}^n$ since

$\nabla \cdot \underset{\sim}{M} \sim \cos\varphi$

$$\Rightarrow \quad \varphi_m^> = b_>^1 \left(\frac{a}{\rho}\right)\cos\varphi$$

$$\varphi_m^< = b_<^1 \left(\frac{\rho}{a}\right)\cos\varphi$$

c) Boundary conditions:

$$B_\rho \Big|_{a-\epsilon}^{a+\epsilon} = 0$$

$$\underbrace{H_\varphi}_{\frac{1}{a}\frac{\partial \varphi_m}{\partial \varphi}} \Big|_{a-\epsilon}^{a+\epsilon} = 0 \quad \Rightarrow \quad \varphi_m \Big|_{a-\epsilon}^{a+\epsilon} = 0$$

d) see previous page for $\Phi_m^{\gtrless}$

matching $\Longrightarrow$

$$\Phi_m^{>}(a) = \Phi_m^{<}(a) \Longrightarrow b_>^1 = b_<^1 \equiv b^1$$

$$B_\rho^{<} = \mu_0\left(-\frac{\partial \Phi_m^<}{\partial \rho} + M_0 \cos\varphi\right)$$

$$B_\rho^{>} = \mu_0\left(-\frac{\partial}{\partial \rho}\Phi_m^{>}\right)$$

$$-b_<^1 \frac{1}{a} + M_0 = b_>^1 \frac{1}{a}$$

$$2b^1 = M_0 a \Rightarrow \boxed{b^1 = \frac{M_0 a}{2}}$$

$$\Phi_m^{>} = \frac{M_0 a^2}{2\rho}\cos\varphi$$

$$\Phi_m^{<} = \frac{M_0}{2}\rho\cos\varphi = \frac{M_0}{2} x$$

e) $\underset{\sim}{B} = \mu_0(\underset{\sim}{H} + \underset{\sim}{M})$

<u>$a > \rho$</u>

$$B_x = \mu_0\left(-\frac{M_0}{2} + M_0\right) = +\frac{\mu_0 M_0}{2}$$

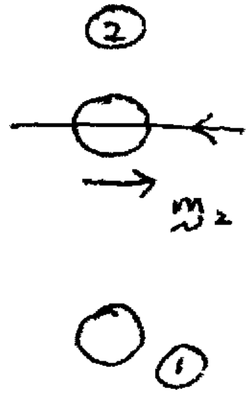
$$B_y = 0$$

<u>$a < \rho$</u>

$$B_\rho = -\mu_0 \frac{\partial \Phi_m^>}{\partial \rho} = \mu_0 \frac{M_0 a^2}{2\rho^2}\cos\varphi$$

$$B_\varphi = -\frac{\mu_0}{\rho}\frac{\partial}{\partial \varphi}\Phi_m^> = \mu_0 \frac{M_0 a^2}{2\rho^2}\sin\varphi$$

(f)



$$B_{1\varphi} \sim \mu_0 \frac{M_0 a^2}{2\rho^2} \sim B_1 \text{ from } (1)$$

$$U_{21} = -\underset{\sim}{m}_2 \cdot \underset{\sim}{B}_1$$

$$= m_2 B_{1\varphi} > 0$$

$$U_{21} = \frac{\text{energy}}{\text{length}} \text{ of } (2) \text{ from } (1)$$

$$m_2 = M_0 \pi a^2 = \frac{\text{magnetic moment}}{\text{length}}$$

$$F_{21} = -\frac{\partial U_{21}}{\partial \rho} = \frac{2\mu_0 M_0 a^2}{2\rho^3} M_0 \pi a^2$$

$$= \frac{2\mu_0 \pi (M_0 a^2)^2}{2\rho^3}$$

$\Rightarrow$ repulsive