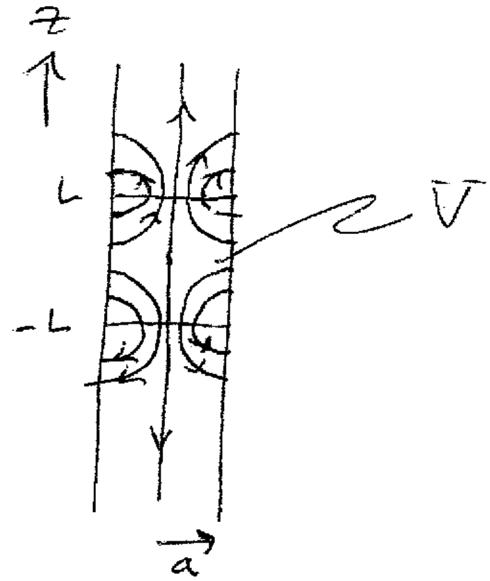
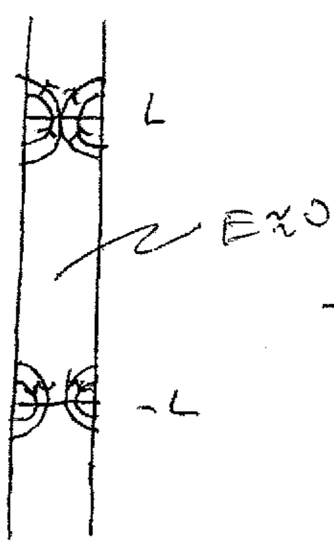
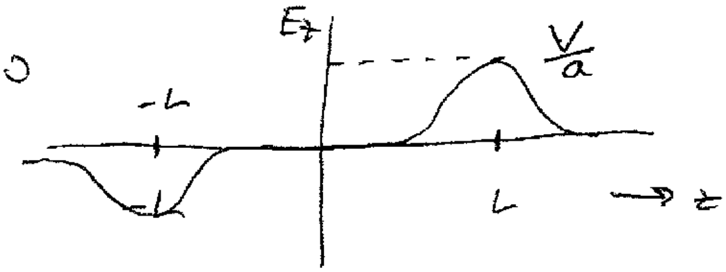
Physics 606 Midterm #1 Solutions

1 a)



b) $a \ll L$





E_z drops ~~off~~ over a scale length "a" for $z > L$

$E_z \sim \frac{V}{a}$ max value for $\rho = 0$

c) Solve Laplace's eqn with $\frac{\partial}{\partial \varphi} = 0$

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho}\Phi + \frac{\partial^2}{\partial z^2}\Phi = 0$$

$\Rightarrow$ oscillatory in z so can match $\Phi(\rho=a, z)$

$$\Phi \sim e^{\pm ikz}$$

$\Rightarrow \cos kz$ since even in z.

$$\Phi \sim \int_0^\infty dk \cos kz \, R_k(\rho)\, g_k$$

$\Rightarrow$ continuous in k since infinite system

Equation for $R_k(\rho)$

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho}R_k - k^2 R_k = 0$$

$\Rightarrow$ modified Bessel equation with $\nu = 0$

$$R_k \sim I_0(k\rho),\ K_0(k\rho)$$

$\Rightarrow K_0 \to \infty$ at $\rho = 0$

$$\Phi = \int_0^\infty dk \cos kz \, I_0(k\rho)\, g_k$$

BC at $\rho = a$

$$\Phi(\rho=a, z) = \int_0^\infty dk \cos kz \, I_0(ka) \, g_k$$

$\Rightarrow$ multiply by $\cos k'z$ and integrate over z

$$V \int_{-L}^{L} dz \cos k'z = \int_0^\infty dk \, I_0(ka) \, g_k \, I$$

$$I = \int_{-\infty}^{\infty} dz \cos kz \cos k'z$$

$$= 2\pi \int_{-\infty}^{\infty} \frac{dz}{2\pi} \frac{\left(e^{ikz} + e^{-ikz}\right)\left(e^{ik'z} + e^{-ik'z}\right)}{4}$$

$$= 2\pi \frac{1}{4}\left[\cancelto{0}{2\delta(k+k')} + 2\delta(k-k')\right]$$

note: $k, k' > 0$

$$V \left.\frac{\sin k'z}{k'}\right|_{-L}^{L} = \pi \, I_0(k'a) \, g_{k'}$$

$$V \, 2 \frac{\sin(k'L)}{k'} = \pi \, I_0(k'a) \, g_{k'}$$

$$\boxed{\Phi = \frac{2V}{\pi} \int_0^\infty dk \, \frac{\sin kL}{k} \cos kz \, \frac{I_0(k\rho)}{I_0(ka)}}$$

~~(2) a) Charge density~~ $\cancel{\rho(\rho, z) = 2q \frac{\delta(z)\delta(\rho)}{2\pi\rho}}$

~~since~~ $\cancel{\int d^3x \, \rho = \int_0^{2\pi} d\phi \int_0^\infty d\rho \int_{-\infty}^{\infty} dz \, \rho \, \frac{\delta(z)\delta(\rho)}{2\pi\rho}}$ ~~...~~

2 a)

$$\nabla^2 \Phi = -\frac{q}{\epsilon_0}\,\delta\left(x-\frac{a}{2}\right)\delta\left(y-\frac{a}{2}\right)\delta(z)$$

Use oscillatory function in x, y directions

$$\Phi = \sum_{\substack{m,n \\ \text{odd}}} \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right) g_{mn}(z)$$

Because of symmetry around $a/2$, m, n are odd integers.

Inserting into Poisson's equation,

$$\sum_{m,n}\left(\frac{\partial^2}{\partial z^2}-K^2\right) g_{mn}\,\sin(k_x x)\sin(k_y y) = -\frac{q}{\epsilon_0}\,\delta(\ )\delta(\ )\delta(\ )$$

with $k_x = \frac{n\pi}{a}$, $k_y = \frac{m\pi}{a}$, $K = (k_x^2+k_y^2)^{\frac{1}{2}}$

Eliminate sum over m, n by multiplying by $\sin\left(\frac{n'\pi x}{a}\right)\sin\left(\frac{m'\pi y}{a}\right)$ and integrating over x, y

$$\frac{a^2}{4}\left(\frac{\partial^2}{\partial z^2}-K^2\right) g_{mn} = -\frac{q}{\epsilon_0}\,\underbrace{\sin\left(\frac{n'\pi}{2}\right)}_{(-1)^{\frac{n'-1}{2}}}\,\underbrace{\sin\left(\frac{m'\pi}{2}\right)}_{(-1)^{\frac{m'-1}{2}}}\,\delta(z)$$

$$\left(\frac{\partial^2}{\partial z^2} - k^2\right) g_{mn} = A_{mn}\ \delta(z)$$

$$A_{mn} = -\frac{q}{\epsilon_0}\frac{4}{a^2}(-1)^{\frac{n-1}{2}}(-1)^{\frac{m-1}{2}}$$

<u>$z > 0$</u>

$$g_{mn} = c_{mn}\, e^{-kz}$$

<u>$z < 0$</u>

$$g_{mn} = c_{mn}\, e^{kz}$$

<u>$z \simeq 0$</u>

$$\frac{\partial^2}{\partial z^2} g_{mn} = A_{mn}\ \delta(z)$$

$\Rightarrow$ integrating

$$\frac{\partial}{\partial z} g_{mn} \Big|_{-\epsilon}^{\epsilon} = A_{mn}$$

$$g_{mn} \Big|_{-\epsilon}^{\epsilon} = 0$$

$$(-k-k)\, c_{mn} = A_{mn}$$

$$c_{mn} = -\frac{1}{2k} A_{mn}$$

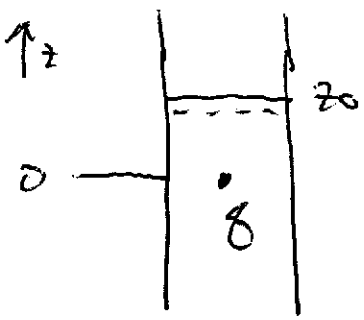
$$\underline{\Phi} = \sum_{\substack{m,n\\ \text{odd}}} \frac{q}{\epsilon_0}\frac{2}{ka^2}(-1)^{\frac{n-1}{2}}(-1)^{\frac{m-1}{2}} e^{-k|z|} \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi x}{a}\right)$$

For large positive z $\Rightarrow$ only smallest k contributes $\Rightarrow m=1, n=1$ $\quad k=\sqrt{2}\left(\frac{\pi}{a}\right)$

$$\Phi \simeq \frac{q}{\epsilon_0}\frac{2}{\sqrt{2}\left(\frac{\pi}{a}\right)a^2}\, e^{-\frac{\sqrt{2}\pi z}{a}} \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{a}\right)$$

$$= \frac{\sqrt{2}\, q}{\epsilon_0 \pi a}\, e^{-\frac{\sqrt{2}\pi z}{a}} \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{a}\right)$$

b) Dielectric added in region $z > z_0 > 0$.



For $q>0$, a negative charge will be induced on the surface of the dielectric at z_0

$\Rightarrow$ attractive force with q

$\Rightarrow$ force on q in positive z direction

c) Change in Φ given by $\delta\Phi$

$$\delta\Phi = \sum_{\substack{m,n\\ \text{odd}}} \delta g_{mn}(z) \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)$$

$\Rightarrow$ continuous across $z=0$ since Φ already produces jump due to q.

$\Rightarrow$ will have jump in $\frac{\partial}{\partial z}\delta\Phi$ across z_0

At surface of dielectric tangential $\underset{\sim}{E}$ is continuous.

$$\Rightarrow \left(\frac{\partial}{\partial x} \delta\Phi + \frac{\partial}{\partial x} \Phi \right)\Bigg|_{z_0-\epsilon}^{z_0+\epsilon} = 0$$

$\Rightarrow$ but $\frac{\partial}{\partial x}\Phi$ continuous at z_0

$$\Rightarrow \frac{\partial}{\partial x} \delta\Phi \Big|_{z_0-\epsilon}^{z_0+\epsilon} = 0$$

$$\Rightarrow \delta g_{mn}(z) \Big|_{z_0-\epsilon}^{z_0+\epsilon} = 0 \quad \cancel{\Rightarrow \delta g_{mn} = \delta C_{mn} e^{-k|z|}}$$

$$\Rightarrow \delta g_{mn} = \delta C_{mn} e^{-k|z-z_0|}$$

At surface of dielectric normal $\underset{\sim}{D}$ is continuous

$$\epsilon \left(\frac{\partial}{\partial z}\Phi + \frac{\partial}{\partial z} \delta\Phi \right)\Bigg|_{z_0-\epsilon}^{z_0+\epsilon} = 0$$

$$\epsilon \left(\frac{\partial}{\partial z}\Phi + \frac{\partial}{\partial z} \delta\Phi \right)\Bigg|_{z_0+\epsilon} = \epsilon_0 \left(\frac{\partial}{\partial z}\Phi + \frac{\partial}{\partial z} \delta\Phi \right)\Bigg|_{z_0-\epsilon}$$

$$\epsilon \left(\frac{\partial}{\partial z} g_{mn} + \frac{\partial}{\partial z} \delta g_{mn} \right)\Bigg|_{z_0+\epsilon} = \epsilon_0 \left(\frac{\partial}{\partial z} g_{mn} + \frac{\partial}{\partial z} \delta g_{mn} \right)\Bigg|_{z_0-\epsilon}$$

d) Once $\delta\Phi$ is known, calculate $\delta E_z = -\frac{\partial}{\partial z}\delta\Phi$

$$F_z = q\, \delta E_z$$