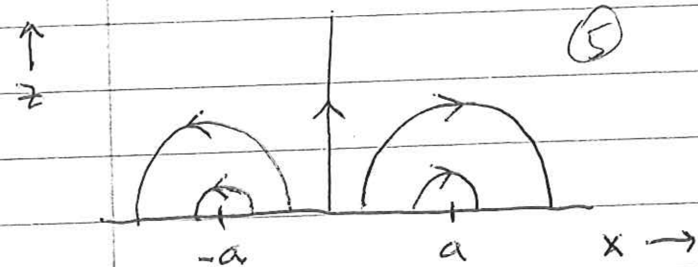
Solutions Midterm # 1

1 a) Sketch $\underset{\sim}{E}$



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Potential falls from V to zero over a distance $\sim \pi a$

$$\Rightarrow E_z \sim \frac{V}{\pi a} \qquad (5)$$

At surface of conductor $\sigma = \epsilon_0 E_n$

$$\Rightarrow \sigma = \frac{\epsilon_0 V}{\pi a}$$

Falloff distance $\sim a$ since

$$\frac{1}{a} \sim \frac{\partial}{\partial x} \sim \frac{\partial}{\partial z} \quad \text{from Laplace's eqn}$$

b) Calculate φ $\Rightarrow \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)\varphi = 0$

$\Rightarrow$ choose oscillatory in x to match specified $\varphi(x, z=0)$.

$\Rightarrow$ decaying in z

$$\varphi(x,z) = \int_0^\infty dk \cos(kx)\, e^{-kz}\, b(k)$$

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$\Rightarrow$ even symmetry in x

Solve for $b(k)$ using BC at $z=0$

$$\varphi(x, z=0) = \int_0^\infty dk \cos kx \; b(k)$$

Operate with $\int_{-\infty}^{\infty} dx \cos(k'x)$

$$V \int_{-a}^{a} dx \cos(k'x) = \int_{-\infty}^{\infty} dx' \underbrace{\cos(k'x)}_{\frac{e^{ik'x}+e^{-ik'x}}{2}} \, \mathrm{Re} \int_0^\infty dk \, e^{ikx} b(k)$$

$$= \frac{2\pi}{2} \int_0^\infty dk \, b(k) \left[\delta(k+k') + \delta(k'-k) \right]$$

$$= \pi \, b(k') \quad \text{since } k>0.$$

$$b(k') = \frac{1}{\pi} V \left. \frac{\sin(k'x)}{k'} \right|_{-a}^{a} = \frac{2V}{\pi} \frac{\sin(k'a)}{k'}$$

$$\varphi = \frac{2V}{\pi} \int_0^\infty dk \, \frac{\sin(ka)}{k} \cos(kx) e^{-kz} \qquad (10)$$

c) Evaluate $E_z = -\frac{\partial \varphi}{\partial z}$ for $x=0$.

$$E_z = -\frac{\partial \varphi}{\partial z} = \frac{2V}{\pi} \int_0^\infty dk \, \sin(ka) \, e^{-kz}$$

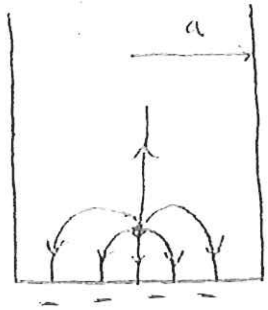
$$= \frac{2V}{\pi} \mathrm{Im} \int_0^\infty dk \, e^{ika} e^{-kz} = \frac{2V}{\pi} \mathrm{Im}\left(\frac{-1}{ia - z} \right)$$

$$E_z = \frac{2V}{\pi} \frac{a}{z^2+a^2}$$

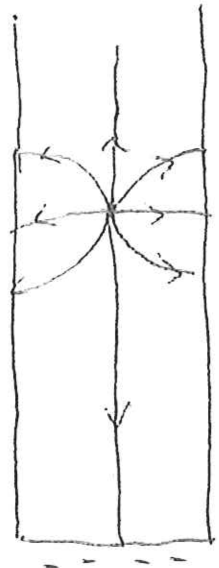
d) $E_z(0,0) = \frac{2V}{\pi a}$, $\sigma = \frac{2\epsilon_0 V}{\pi a}$

$\Rightarrow$ consistent with (1a).

2 a)



$b \ll a$



$b \gg a$

b) $b \ll a$

Force on q downward since negative charge induced on base (for $q > 0$).

To lowest order can neglect the sides of the cylinder $\Rightarrow$ charge sees an infinite plane conductor

$\Rightarrow$ method of Images

$$\underset{\sim}{E}_{ext} \simeq -\frac{q}{4b^2}\,\hat{z}\,\frac{1}{4\pi\epsilon_0}$$

$$\boxed{\underset{\sim}{F} \simeq -\frac{q^2}{4b^2}\,\hat{z}\,\frac{1}{4\pi\epsilon_0}}$$

c) $\Phi \to 0$ at $\rho = a$ $\Rightarrow J_0\left(x_{0i}\frac{\rho}{a}\right)$ as basis functions

$$\Phi = \sum_i J_0\left(x_{0i}\frac{\rho}{a}\right) g_i(z)$$

$$Q = 2q\,\frac{\delta(z-b)}{2\pi\rho}\,\delta(\rho) \Rightarrow \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dz \int_0^{\infty} d\rho\, \rho Q = q$$

d) $$\nabla^2 \Phi = -Q\,\frac{1}{\epsilon_0}$$

$$k_i \equiv \frac{x_{0i}}{a}$$

$$\sum_i \left(\frac{\partial^2}{\partial z^2} g_i - k_i^2 g_i\right) J_0(k_i\rho) = -\frac{2q}{2\pi\rho}\,\frac{\delta(z-b)\,\delta(\rho)}{\epsilon_0}$$

operate with $\int_0^a d\rho\, \rho\, J_0(k_j\rho)$

$$\frac{\partial^2}{\partial z^2} g_j - k_j^2 g_j = -\frac{q\, \delta(z-b)\, J_0(0)}{\pi \frac{a^2}{2} J_1^2(x_{0j})\, \epsilon_0}$$

$$\equiv A_j\, \delta(z-b)$$

$z \neq b$

$z > b \qquad g_i = B_j\, e^{-k_i z} \sinh (k_i b)$

$z < b \qquad g_i = B_j\, e^{-k_i b} \sinh (k_i z)$

$\Rightarrow$ so that $\varphi \to 0$ at $z = 0$ and $z \to \infty$

jump condition at $z = b$

$$\left.\frac{\partial g_i}{\partial z}\right|_{b-\epsilon}^{b+\epsilon} = A_j$$

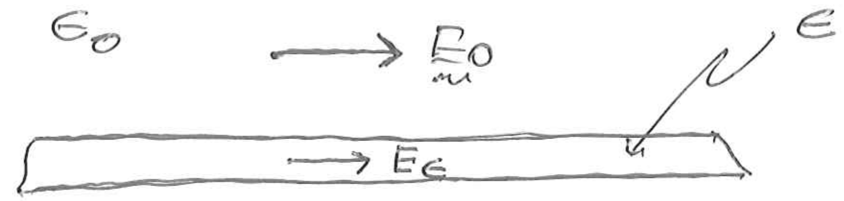
$$-B_i k_i \left(+\cancel{k_i} e^{-k_i b} \sinh(k_i b) + e^{-k_i b} \cosh k_i b \right] = A_j$$

$$B_i = \frac{A_j}{-k_i} \cancel{e^{k_i b}} \frac{1}{\cancel{e^{k_i b}}} = -\frac{A_j}{k_i}$$

$$\varphi = + \sum_i \frac{J_0\left(\frac{x_{0i} \rho}{a}\right)}{\left(\frac{x_{0i}}{a}\right) \epsilon_0} \frac{q}{\pi a^2} \frac{1}{J_1^2(x_{0i})} e^{-\frac{x_{0i}}{a} z_>} \sinh\left(\frac{x_{0i}}{a} z_<\right.$$

(3)

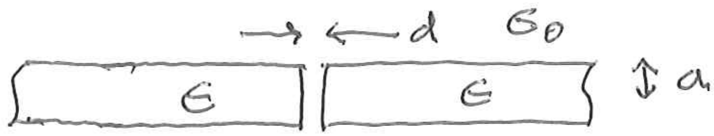
a)



Because the system is homogeneous in z, the electric field remains in the z direction. Since $\nabla \times \underline{E} = 0$, the tangential $\underline{E}$ is conserved

$$\Rightarrow \quad E_\epsilon = E_0$$

b)



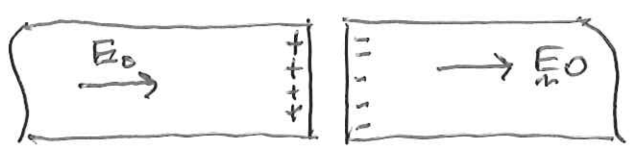
Since no free charge, D_z is continuos

$$\epsilon_0 E_d = \epsilon E_0$$

$$E_d = \frac{\epsilon}{\epsilon_0} E_0 > E_0$$

$\Rightarrow$ E_d is greater than in the dielectric

c)



d) Energy density in a dielectric is $\frac{1}{2}\epsilon E^2$

In the gap, $W_d = \frac{1}{2}\epsilon_0\left(\frac{\epsilon^2}{\epsilon_0^2}E_0^2\right)\pi a^2 d$

Energy change

$$\Delta W = \frac{1}{2}\pi a^2 d\, E_0^2\left(\frac{\epsilon^2}{\epsilon_0} - \epsilon\right)$$

$$\text{force} = -\frac{\partial}{\partial d}\Delta W = -\frac{1}{2}\pi a^2 E_0^2 \frac{\epsilon}{\epsilon_0}(\epsilon - \epsilon_0) \quad \text{attractive}$$