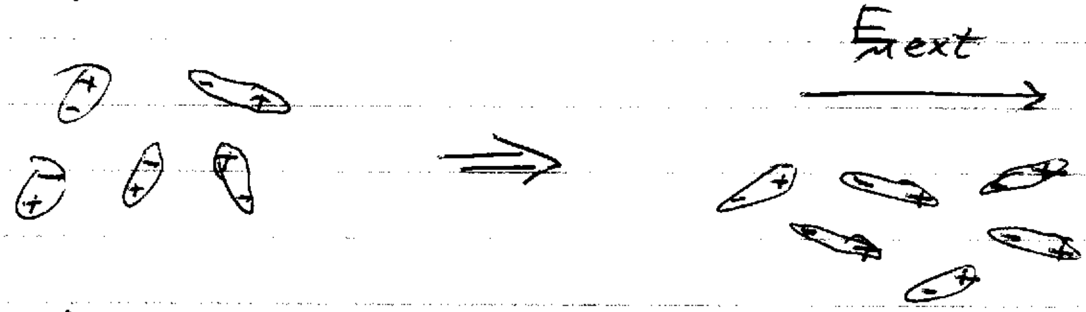
Electric fields in dielectric materials

We have previously only considered the response of conductors to electric fields. We now consider the response of materials that have no free charge to electric fields
$\Rightarrow$ dielectric materials

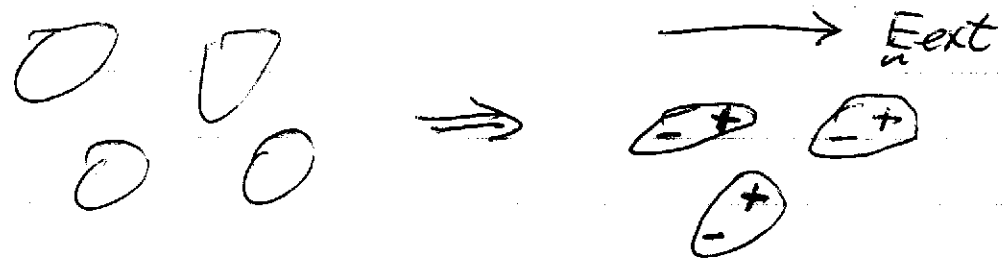
When an electric field is applied to a dielectric, individual molecules respond by producing a local dipole moment. There are two general cases: polar and non-polar molecules

polar molecules: molecules that are initially polar respond by trying to orient with respect to $\underline{E}$



Since $W = -\underline{p}_m \cdot \underline{E}_{ext}$, the molecules tend to orient with $\underline{p}_m$ along $\underline{E}$. Alignment is incomplete $\Rightarrow$ thermal motion interferes with alignment.

non-polar molecules : non-polar molecules are nearly symmetric, but $\underset{\sim}{E}$ induces a distortion of the electric charge of the molecule that produces local dipoles



It is because $\underset{\sim}{E}$ is the dipole component of the potential that $\underset{\sim}{E}$ induces a dipole moment in the molecules

$\Rightarrow$ dipoles interact with dipoles.

In either case the dielectric gains a net dipole moment per unit volume

$$\underset{\sim}{P}(\underset{\sim}{x}) = \sum_i N_i(\underset{\sim}{x}) \langle \underset{\sim}{P}_i \rangle$$

where $N_i(\underset{\sim}{x})$ is the number of molecules per unit volume of the ith type and $\langle \underset{\sim}{P}_i \rangle$ is the average dipole moment of those molecules.

The potential due to the dielectric in an infinitesimal volume $d\underset{\sim}{x}'$ is

$$\delta \varphi(\underset{\sim}{x}) = \frac{1}{4\pi\epsilon_0}\left[\frac{\rho(\underset{\sim}{x}')}{|\underset{\sim}{x}-\underset{\sim}{x}'|} + \frac{\underset{\sim}{P}(\underset{\sim}{x}')\cdot(\underset{\sim}{x}-\underset{\sim}{x}')}{|\underset{\sim}{x}-\underset{\sim}{x}'|^3}\right]d\underset{\sim}{x}'$$

with $\rho(\underset{\sim}{x}')$ the local charge density. Since

$$-\nabla\frac{1}{|\underset{\sim}{x}-\underset{\sim}{x}'|} = \nabla'\frac{1}{|\underset{\sim}{x}-\underset{\sim}{x}'|} = \frac{\underset{\sim}{x}-\underset{\sim}{x}'}{|\underset{\sim}{x}-\underset{\sim}{x}'|^3}$$

~~Note that~~

so that

$$\varphi(\underset{\sim}{x}) = \int d\underset{\sim}{x}'\left[\frac{\rho(\underset{\sim}{x}')}{|\underset{\sim}{x}-\underset{\sim}{x}'|} + \underset{\sim}{P}(x')\cdot\nabla'\frac{1}{|\underset{\sim}{x}-\underset{\sim}{x}'|}\right]\frac{1}{4\pi\epsilon_0}$$

$\Rightarrow$ can integrate by parts with respect to ∇' with $P \to 0$ at the boundaries,

$$\varphi(\underset{\sim}{x}) = \int d\underset{\sim}{x}'\ \frac{\rho(\underset{\sim}{x}') - \nabla'\cdot\underset{\sim}{P}(\underset{\sim}{x}')}{|\underset{\sim}{x}-\underset{\sim}{x}'|}\ \frac{1}{4\pi\epsilon_0}$$

Thus, the total local charge density, including the dielectric response is

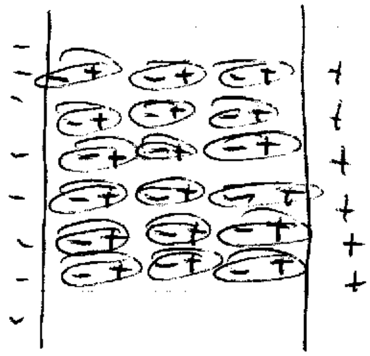
$$\rho_{tot} = \rho - \nabla\cdot\underset{\sim}{P}$$

and

$$\nabla\cdot\underset{\sim}{E} = \frac{1}{\epsilon_0}\left(\rho - \nabla\cdot\underset{\sim}{P}\right)$$

$-\nabla \cdot \underset{\sim}{P}(\underset{\sim}{x})$ represents the excess charge associated with the dipole response to the electric field

$\underset{\sim}{E}_{ext} \longrightarrow$



There is no excess charge when $\underset{\sim}{P}$ is uniform since the charge associated with individual dipoles cancel.

Can rewrite Poisson's eqn as

$$\nabla \cdot \underset{\sim}{D} = \rho(\underset{\sim}{x})$$

$$\underset{\sim}{D} = \epsilon_0 \underset{\sim}{E} + \underset{\sim}{P}(\underset{\sim}{x})$$

with $\underset{\sim}{D}(\underset{\sim}{x})$ the electric displacement. Note that $\underset{\sim}{D}$ is a defined quantity. It has no actual physical existence

$\Rightarrow$ $\underset{\sim}{D}$ is the field from the free charge

$\Rightarrow$ $\underset{\sim}{E}$ is the measureable field, which arises from all the charge

Can typically write

$$\underset{\sim}{P} = \epsilon_0 \chi_e \underset{\sim}{E}$$

$\Rightarrow$ the polarization increases linearly with the strength of $\underset{\sim}{E}$

$\Rightarrow$ must fail if $\underset{\sim}{E}$ is too large

$\Rightarrow$ χ_e = electric susceptibility

Can then write

$$\underset{\sim}{D} = \epsilon \underset{\sim}{E}$$

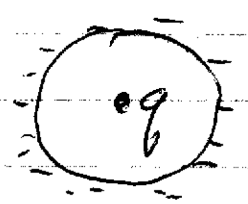
with

$$\epsilon = \epsilon_0 (1 + \chi_e) = \text{dielectric constant}$$

In a uniform dielectric, ϵ = const. and

$$\nabla \cdot \underset{\sim}{E} = \frac{\rho}{\epsilon} < \frac{\rho}{\epsilon_0}$$

so charge is shielded by the dielectric ($\underset{\sim}{E}$ is reduced).

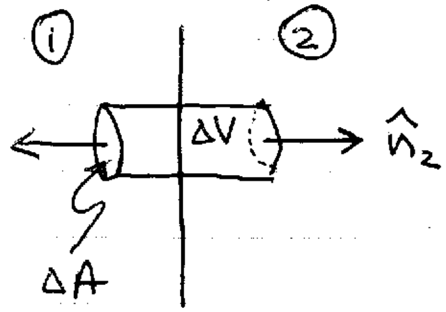


For ϵ large, very large dipole response and strong shielding.

For uniform dielectric can solve for $\underset{\sim}{E}$ and φ as before with ϵ replacing ϵ_0.

<u>Boundary conditions</u>

In most systems have boundaries either between a dielectric and a vacuum or between two dielectrics. What are the boundary conditions?



Gaussian pillbox

$$\nabla \cdot \underset{\sim}{D} = \rho$$

$$(\underset{\sim}{D}_2 - \underset{\sim}{D}_1) \cdot \hat{n}_2 = \sigma_{12}$$

with σ_{12} the free surface charge

The normal component of $\underset{\sim}{D}$ is continuous if $\sigma_{12} = 0$.

$\Rightarrow$ $\underset{\sim}{E} \cdot \hat{n}$ is not continuous since there is always a surface charge associated with the polarization

$$\rho_{pol} = -\nabla \cdot \underset{\sim}{P}$$

$$\int_{\Delta V} d\underset{\sim}{x}\, \rho_{pol} = \sigma_{pol}\, \Delta A = -(\underset{\sim}{P}_2 - \underset{\sim}{P}_1) \cdot \hat{n}_2\, \Delta A$$

$$\sigma_{pol} = -(\underset{\sim}{P}_2 - \underset{\sim}{P}_1) \cdot \hat{n}_2$$

$\Rightarrow$ calculate $\underset{\sim}{P}$ from $\underset{\sim}{E}$

$\Rightarrow$ e.g. $\underset{\sim}{P}_2 = \epsilon_0 \chi_{e2} \underset{\sim}{E}_2$

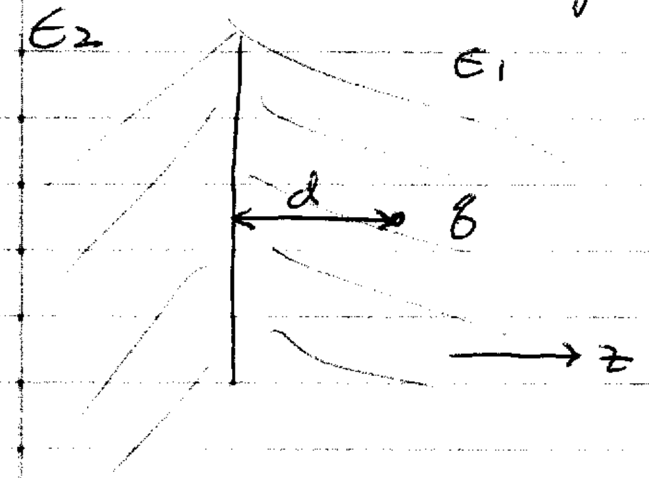
$$\underset{\sim}{P}_1 = \epsilon_0 \chi_{e1} \underset{\sim}{E}_1$$

Also, unlike a conductor $\hat{n} \times \underset{\sim}{E}$ is not zero. From $\nabla \times \underset{\sim}{E} = 0$

$$\left(\underset{\sim}{E}_2 - \underset{\sim}{E}_1\right) \times \hat{n}_2 = 0$$

$\Rightarrow$ the tangential component of $\underset{\sim}{E}$ is continuous.

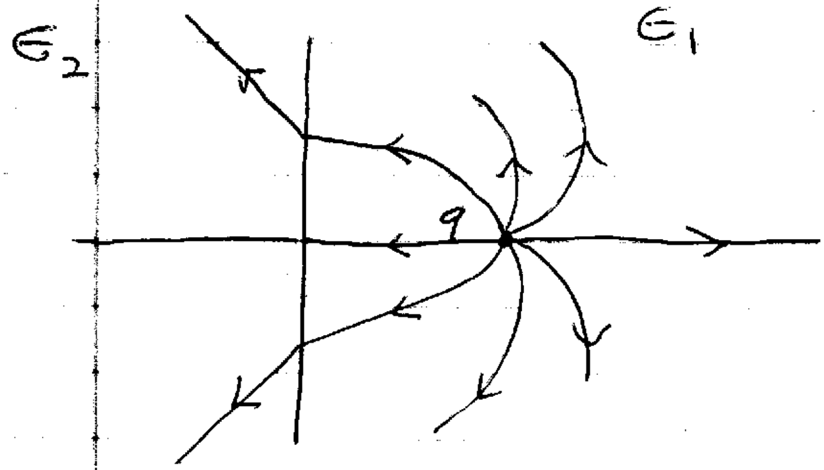
<u>Example</u>: Charge embedded in dielectric half-space



Take $q > 0$

Qualitative description:

$\epsilon_2 \gg \epsilon_1$:

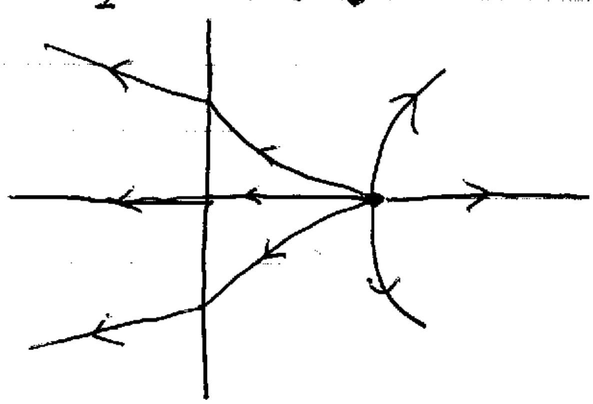


Negative charge will be induced on surface to shield $\underset{\sim}{E}$ due to q from ϵ_2.

$\Rightarrow$ induced charge negative for $q > 0$.

What happens in the limit $\epsilon_2 \longrightarrow \infty$?
Total charge on surface?
$\underset{\sim}{E}$ within ϵ_2?

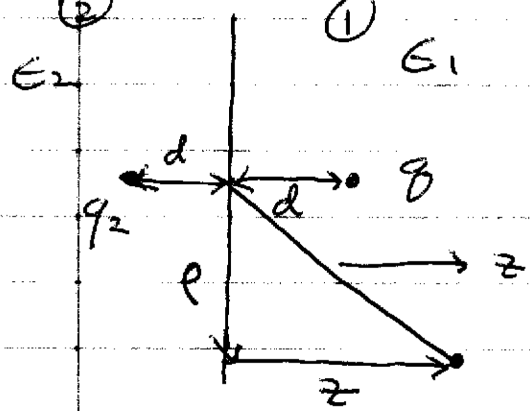
$\epsilon_2 \ll \epsilon_1$:



Induce charge same as q.
Medium ϵ_2 tries to reduce the shielding of q from ϵ_1.

Quantitative solution:

Use image charge in ϵ_2 to find Φ in region 1



In region 1

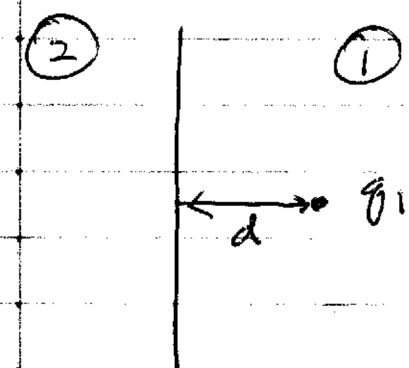
$$\Phi_1 = \frac{1}{4\pi\epsilon_1}\left(\frac{q}{r_1} + \frac{q_2}{r_2}\right)$$

$$r_1^2 = (z-d)^2 + \rho^2$$

$$r_2^2 = (z+d)^2 + \rho^2$$

Note: doesn't matter whether write q_2/ϵ_1 or q_2/ϵ_2 since q_2 is a free variable

In region 2



In region 2 have effective charge q_1 at location $z=d$

$$\Phi_2 = \frac{1}{4\pi\epsilon_2}\,\frac{q_1}{r_1}$$

Two unknowns q_1, q_2 but two BCs, normal $\underline{D}$ continuous and tangential $\underline{E}$ continuous.

Normal $\underset{\sim}{D}$: at $z=0$

$$D_z^1 = D_z^2 \implies \epsilon_1 E_z^1 = \epsilon_2 E_z^2$$

$$\epsilon_1 \frac{1}{4\pi\epsilon_1}\left[-\frac{1}{2}\frac{q}{r_1^3}2(z-d) - \frac{1}{2} q_2 \frac{1}{r_2^3} 2(z+d)\right]\Bigg|_{z=0}$$

$$= \frac{\epsilon_2}{4\pi\epsilon_2} q_1 \left(-\frac{1}{2}\frac{2(z-d)}{r_1^3}\right)\Bigg|_{z=0}$$

$$\frac{\partial}{\partial z}\frac{1}{r_1} = \frac{\partial}{\partial z}\frac{1}{[(z-d)^2+\rho^2]^{1/2}} = -\frac{1}{2}\frac{2(z-d)}{r_1^3}$$

At $z=0$, $r_1 = r_2$.

$$q d - q_2 d = q_1 d$$

$$q - q_2 = q_1$$

Tangential $\underset{\sim}{E}$: at $z=0$

$$E_\rho^1 = E_\rho^2$$

$$\frac{1}{\epsilon_1}\left[-\frac{1}{2}\frac{q}{r_1^3}2\rho - \frac{1}{2} q_2 \frac{2\rho}{r_2^3}\right]\Bigg|_{z=0} = \frac{1}{\epsilon_2}\left[-\frac{1}{2}\frac{q_1}{r_1^3}2\rho\right]\Bigg|_{z=0}$$

$$\frac{1}{\epsilon_1}\left[-q - q_2\right] = \frac{1}{\epsilon_2}(-q_1)$$

$$q + q_2 = \frac{\epsilon_1}{\epsilon_2} q_1 = \frac{\epsilon_1}{\epsilon_2}(q - q_2)$$

$$q_2\left(1+\frac{\epsilon_1}{\epsilon_2}\right) = q\left(\frac{\epsilon_1}{\epsilon_2}-1\right)$$

$$q_2 = q\,\frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2}$$

$$q_1 = \frac{2\epsilon_2}{\epsilon_1+\epsilon_2}\,q$$

$$\Phi_1 = \frac{1}{4\pi\epsilon_1}\left[\frac{q}{r_1} + q\,\frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2}\,\frac{1}{r_2}\right]$$

$$\Phi_2 = \frac{1}{4\pi}\,\frac{2q}{\epsilon_1+\epsilon_2}\,\frac{1}{r_1}$$

Polarization charge: $\quad \underset{\sim}{D} = \epsilon \underset{\sim}{E}$

$$\sigma_{pol} = -\left(\underset{\sim}{P}_2-\underset{\sim}{P}_1\right)\cdot\hat{n}_2$$

$$= -\epsilon_0\left(\chi_{e2}\underset{\sim}{E}_2 - \chi_{e1}\underset{\sim}{E}_1\right)\cdot\hat{n}_2$$

$$= -\epsilon_0\left(\frac{\chi_{e2}}{\epsilon_2}-\frac{\chi_{e1}}{\epsilon_1}\right)\underset{\sim}{D}_2\cdot\hat{n}_2$$

$\Longrightarrow$ since D_z is continuous

$$E_{z2} = -\frac{\partial}{\partial z}\Phi_2\Big|_{z=0} = -\frac{1}{4\pi}\,\frac{2q}{\epsilon_1+\epsilon_2}\left[-\frac{1}{2}\,\frac{1}{r_1^3}\,2(z-d)\right]\Big|_{z=0}$$

$$= -\frac{1}{2\pi}\,\frac{qd}{(\rho^2+d^2)^{3/2}}\,\frac{1}{\epsilon_1+\epsilon_2}$$

$D_{z2} = \epsilon_2 E_{z2} \quad$ Note $\underset{\sim}{D}_2\cdot\hat{\underset{\sim}{n}}_2 = -D_{2z}$

$$\chi_e = \frac{\epsilon}{\epsilon_0} - 1$$

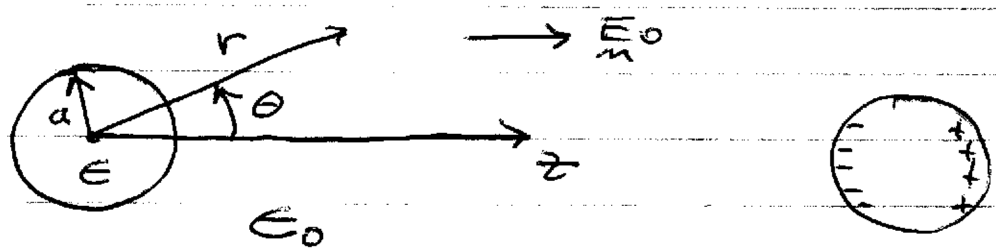
$$\sigma_{pol} = -\epsilon_0 \underbrace{\left(\frac{\chi_{e2}}{\epsilon_2} - \frac{\chi_{e1}}{\epsilon_1} \right)}_{\underbrace{\frac{\epsilon_2}{\epsilon_0 \epsilon_2} - \frac{1}{\epsilon_2} - \frac{\epsilon_1}{\epsilon_0 \epsilon_1} + \frac{1}{\epsilon_1}}_{\underbrace{\frac{1}{\epsilon_1} - \frac{1}{\epsilon_2}}_{\frac{\epsilon_2 - \epsilon_1}{\epsilon_1 \epsilon_2}}}} \left(+ \frac{1}{2\pi} \frac{q d}{(\rho^2 + d^2)^{3/2}} \frac{\epsilon_2}{\epsilon_1 + \epsilon_2} \right)$$

$$\sigma_{pol} = -\frac{1}{2\pi} q \frac{d}{(\rho^2 + d^2)^{3/2}} \frac{\epsilon_0 (\epsilon_2 - \epsilon_1)}{\epsilon_1 (\epsilon_2 + \epsilon_1)}$$

Check: For $\epsilon_2 > \epsilon_1 \implies \sigma_{pol} < 0$

For $\epsilon_2 < \epsilon_1 \implies \sigma_{pol} > 0$

example Dielectric sphere in uniform $\underset{\sim}{E_0}$



$$\nabla \times \underset{\sim}{E} = 0 \quad \Rightarrow \quad \underset{\sim}{E} = -\nabla \varphi$$

$$\underset{\sim}{D} = \epsilon \underset{\sim}{E} = -\epsilon \nabla \varphi$$

$$\nabla \cdot \underset{\sim}{D} = 0 \quad \Rightarrow \quad \nabla \cdot (\epsilon \nabla \varphi) = 0$$

$\Rightarrow$ away from boundary of sphere

$$\nabla^2 \varphi = 0$$

$\Rightarrow$ azimuthal symmetry

Inside sphere:

$$\varphi_< = \sum_n a_n r^n P_n(\cos\theta)$$

Outside sphere:

$$\varphi_> = -E_0 \overbrace{\underbrace{r\cos\theta}_{P_1(\cos\theta)}}^{z} + \sum_n b_n \frac{1}{r^{n+1}} P_n(\cos\theta)$$

Boundary conditions at $r = a$:

E_θ continuous

D_r continuous

Radial $\underset{\sim}{D}$:

$$\epsilon \frac{\partial \varphi_<}{\partial r}\Big|_{r=a} = \epsilon_0 \frac{\partial \varphi_>}{\partial r}\Big|_{r=a}$$

$$\frac{\epsilon}{\epsilon_0} \sum_n n\, a^{n-1} P_n(\cos\theta)\, a_n = -E_0 P_1(\cos\theta) - \sum_n b_n \frac{(n+1)}{a^{n+2}} P_n(\cos\theta)$$

For $n = 1$,

$$\frac{\epsilon}{\epsilon_0} a_1 = -E_0 - b_1 \frac{2}{a^3}$$

For $n \neq 1$

$$\frac{\epsilon}{\epsilon_0} n\, a^{n-1} a_n = -b_n \frac{(n+1)}{a^{n+2}}$$

Tangential $\underset{\sim}{E}$:

$$\frac{\partial}{\partial\theta}\varphi_<\Big|_{r=a} = \frac{\partial}{\partial\theta}\varphi_>\Big|_{r=a}$$

$n=1$,

$$a_1 a = -E_0 a + b_1 \frac{1}{a^2}$$

$$\Rightarrow a_1 = -E_0 + b_1/a^3$$

$n \neq 1$,

$$a_n a^n = b_n \frac{1}{a^{n+1}}$$

For $n \neq 1$, two independent expressions relating a_n to b_n

$\Rightarrow$ incompatible

$\Rightarrow a_n = b_n = 0$

For $n = 1$,

$$\frac{\epsilon}{\epsilon_0}\left(-E_0 + \frac{b_1}{a^3}\right) = -E_0 - b_1 \frac{2}{a^3}$$

$$b_1 = E_0\left(\frac{\epsilon}{\epsilon_0} - 1\right) \frac{a^3}{\frac{\epsilon}{\epsilon_0} + 2}$$

$$b_1 = E_0 \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} a^3$$

$$a_1 = -E_0 + E_0 \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0}$$

$$= -E_0 \frac{3\epsilon_0}{\epsilon + 2\epsilon_0}$$

<u>Inside sphere</u>:

$$\Phi_< = -E_0 \frac{3\epsilon_0}{\epsilon + 2\epsilon_0} z$$

$$E_z = E_0 \frac{3\epsilon_0}{\epsilon + 2\epsilon_0}$$

$\Rightarrow$ uniform electric field

$\Rightarrow$ reduction of E_z inside sphere.

Stored energy with dielectrics

We previously found that the energy associated with a charge distribution $\rho(\underset{\sim}{x})$ is

$$W = \frac{1}{2} \int d\underset{\sim}{x}\, \rho(\underset{\sim}{x})\, \Phi(\underset{\sim}{x})$$

Want to calculate the corresponding energy in the presence of dielectrics.

Consider a dielectric medium $\epsilon(\underset{\sim}{x})$ with no charge. We can calculate the work necessary to assemble charge from infinity. To bring charge $\delta\rho(\underset{\sim}{x})$ from infinity requires work

$$\delta W = \int d\underset{\sim}{x}\, \delta\rho(\underset{\sim}{x})\, \Phi(\underset{\sim}{x}),$$

where $\Phi(\underset{\sim}{x})$ is the potential produced by the charge $\rho(\underset{\sim}{x})$ already assembled. Note that $\delta\rho(\underset{\sim}{x})$ does not include the induced charge. This enters only through the modification of $\Phi(\underset{\sim}{x})$. Since $\nabla \cdot \underset{\sim}{D} = \rho$,

$$\nabla \cdot \delta \underset{\sim}{D} = \delta\rho,$$

and

$$\delta \bar{W} = \int d\underset{\sim}{x}\, \varphi(\underset{\sim}{x})\, \nabla \cdot \delta \underset{\sim}{D}$$

$$= \int d\underset{\sim}{x}\, \underset{\sim}{E} \cdot \delta \underset{\sim}{D}$$

where boundary contributions are neglected. We assume for simplicity that the dielectric is linear so that

$$\underset{\sim}{D} = \epsilon \underset{\sim}{E}$$

and

$$\underset{\sim}{E} \cdot \underset{\sim}{D} = \epsilon \underset{\sim}{E} \cdot \underset{\sim}{E}$$

$$\delta(\underset{\sim}{E} \cdot \underset{\sim}{D}) = 2 \epsilon\, \delta \underset{\sim}{E} \cdot \underset{\sim}{E} = 2\, \delta \underset{\sim}{D} \cdot \underset{\sim}{E}$$

so

$$\underset{\sim}{E} \cdot \delta \underset{\sim}{D} = \frac{1}{2}\, \delta \left[\epsilon |\underset{\sim}{E}|^2 \right]$$

$$\delta \bar{W} = \int d\underset{\sim}{x}\, \frac{1}{2}\, \delta \left[\epsilon |\underset{\sim}{E}|^2 \right]$$

Integrating $\underset{\sim}{E}$ from zero to the final value,

$$\bar{W} = \frac{1}{2} \int d\underset{\sim}{x}\, \epsilon(\underset{\sim}{x}) |\underset{\sim}{E}|^2$$

Thus, for a given $\underset{\sim}{E}$ there is more energy stored in the presence of a dielectric since $\epsilon > \epsilon_0$. Considering a virtual displacement with this expression can be used to determine forces on materials.

Can also explore forces acting on dielectrics by calculating the energy associated with the variation of $\epsilon(\underset{\sim}{x})$.
Thus, consider the free charge ρ to be fixed and change the dielectric from ϵ_0 to $\epsilon(\underset{\sim}{x})$. In the initial state

$$W_0 = \frac{1}{2}\int d\underset{\sim}{x}\; \underset{\sim}{E}_0 \cdot \underset{\sim}{D}_0 .$$

In the final state have

$$W = \frac{1}{2}\int d\underset{\sim}{x}\; \underset{\sim}{E} \cdot \underset{\sim}{D}$$

so

$$\Delta W = W - W_0 = \frac{1}{2}\int d\underset{\sim}{x}\left(\underset{\sim}{E}\cdot\underset{\sim}{D} - \underset{\sim}{E}_0\cdot\underset{\sim}{D}_0\right)$$

$$= \frac{1}{2}\int d\underset{\sim}{x}\left(\Phi\, \nabla\cdot\underset{\sim}{D} - \Phi_0\, \nabla\cdot\underset{\sim}{D}_0\right)$$

after integration by parts. Since $\rho(\underset{\sim}{x})$ is not changed, $\nabla\cdot\underset{\sim}{D} = \nabla\cdot\underset{\sim}{D}_0$. Thus, we can exchange $\underset{\sim}{D}$ and $\underset{\sim}{D}_0$,

$$\Delta W = \frac{1}{2}\int d\underset{\sim}{x}\left(\Phi\, \nabla\cdot\underset{\sim}{D}_0 - \Phi_0\, \nabla\cdot\underset{\sim}{D}\right)$$

$$= \frac{1}{2}\int d\underset{\sim}{x}\left(\underset{\sim}{E}\cdot\underset{\sim}{D}_0 - \underset{\sim}{E}_0\cdot\underset{\sim}{D}\right)$$

or

$$\Delta W = \frac{1}{2} \int_V d\underset{\sim}{x}\ (\epsilon_0 - \epsilon)\ \underset{\sim}{E} \cdot \underset{\sim}{E}_0$$

with V the volume over which ϵ was changed. Since $\epsilon = \epsilon_0(1+\chi_e)$,

$$\Delta W = -\frac{1}{2} \int d\underset{\sim}{x}\ \epsilon_0 \chi_e\ \underset{\sim}{E} \cdot \underset{\sim}{E}_0$$

but $\underset{\sim}{P} = \epsilon_0 \chi_e \underset{\sim}{E}$ so

$$\Delta W = -\frac{1}{2} \int d\underset{\sim}{x}\ P \cdot \underset{\sim}{E}_0$$

with $\underset{\sim}{P}(\underset{\sim}{x})$ the dipole moment per unit volume. The energy density ~~re~~ of the dielectric in the external field is

$$-\frac{1}{2} \underset{\sim}{P} \cdot \underset{\sim}{E}_0 \ .$$

In which direction will a dielectric in a non-uniform medium move?

Note the $\frac{1}{2}$ in front of the energy density. A fixed dipole has no factor of $\frac{1}{2}$. The factor of $\frac{1}{2}$ arises because of the work needed to polarize the dielectric.