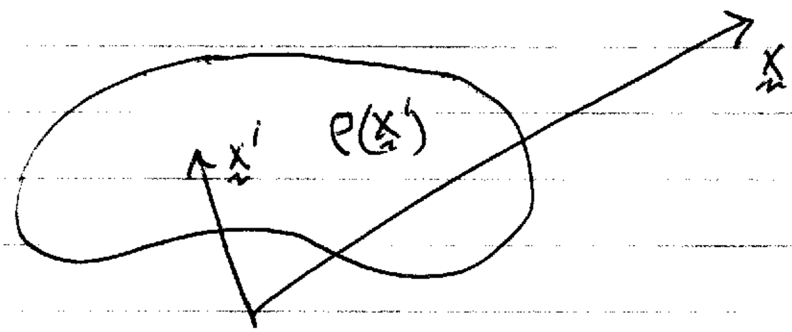
Multipole expansion of the potential from charge distributions

Consider an arbitrary localized region of charge $\rho(\mathbf{x}')$



Want to calculate the potential $\Phi(\mathbf{x})$ outside and far from the charges

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int d\mathbf{x}' \, \rho(\mathbf{x}') \frac{1}{|\mathbf{x} - \mathbf{x}'|}$$

with $|\mathbf{x}| \equiv x \gg |\mathbf{x}'| = x'$

$$|\mathbf{x} - \mathbf{x}'| = \left(x^2 + x'^2 - 2\,\mathbf{x}\cdot\mathbf{x}'\right)^{\frac{1}{2}}$$

$$= x\left(1 + \frac{x'^2}{x^2} - 2\frac{\mathbf{x}\cdot\mathbf{x}'}{x^2}\right)^{\frac{1}{2}}$$

$$\frac{1}{(1+\epsilon)^{\frac{1}{2}}} \simeq 1 - \frac{1}{2}\epsilon + \frac{1}{2}\frac{3}{4}\epsilon^2$$

$$\frac{1}{|\underset{\sim}{x}-\underset{\sim}{x}'|} \simeq \frac{1}{x}\left[1+\frac{\underset{\sim}{x}\cdot\underset{\sim}{x}'}{x^2}-\frac{1}{2}\frac{x'^2}{x^2}+\frac{3}{8}\,4\frac{(\underset{\sim}{x}\cdot\underset{\sim}{x}')^2}{x^4}\right]$$

$$\varphi(\underset{\sim}{x}) \simeq \frac{1}{4\pi\epsilon_0}\left[\int d\underset{\sim}{x}'\,\frac{\rho(\underset{\sim}{x}')}{x}+\int d\underset{\sim}{x}'\,\frac{\rho(\underset{\sim}{x}')\,\underset{\sim}{x}'\cdot\underset{\sim}{x}}{x^3}\right.$$

$$\left.+\frac{1}{2}\int d\underset{\sim}{x}'\left(3\frac{(\underset{\sim}{x}\cdot\underset{\sim}{x}')^2}{x^4}-\frac{x'^2}{x^2}\right)\frac{\rho(\underset{\sim}{x}')}{x}\right]$$

$$=\frac{1}{4\pi\epsilon_0}\left[\frac{q}{x}+\frac{\underset{\sim}{p}\cdot\underset{\sim}{x}}{x^3}+\frac{1}{2}\frac{1}{x^5}\underset{\approx}{Q}:\underset{\sim}{x}\underset{\sim}{x}\right] + \dots$$

$$q=\int d\underset{\sim}{x}'\,\rho(\underset{\sim}{x}')$$

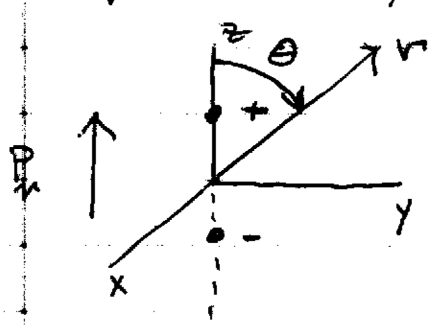
$\underset{\sim}{p}=\int d\underset{\sim}{x}'\,\rho(\underset{\sim}{x}')\,\underset{\sim}{x}'$ = dipole moment

$Q_{ij}=\int d\underset{\sim}{x}'\,\rho(\underset{\sim}{x}')\left[3x_i'x_j'-x'^2\delta_{ij}\right]$

= quadrupole moment

$\Rightarrow$ for large x the lowest non-zero moment dominates

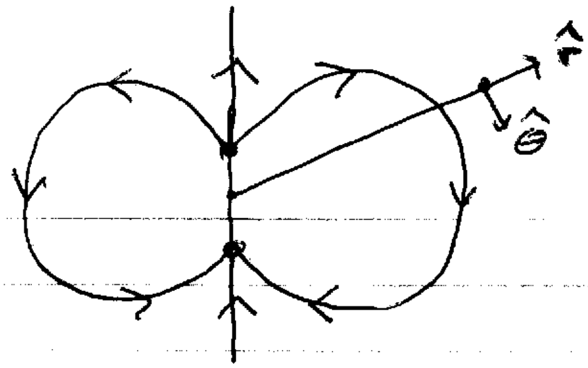
<u>Dipole</u>: pointed along z



$$\varphi=\frac{p_z\cos\theta}{r^2}\,\frac{1}{4\pi\epsilon_0}$$

$$E_r=-\frac{\partial\varphi}{\partial r}=\frac{2p_z\cos\theta}{r^3}\,\frac{1}{4\pi\epsilon_0}$$

$$E_\theta=-\frac{1}{r}\frac{\partial}{\partial\theta}\varphi=\frac{p_z\sin\theta}{r^3}\,\frac{1}{4\pi\epsilon_0}$$



For arbitrary orientation: $\Phi = -\frac{1}{4\pi\epsilon_0}\, \mathbf{p}\cdot\nabla\frac{1}{x}$

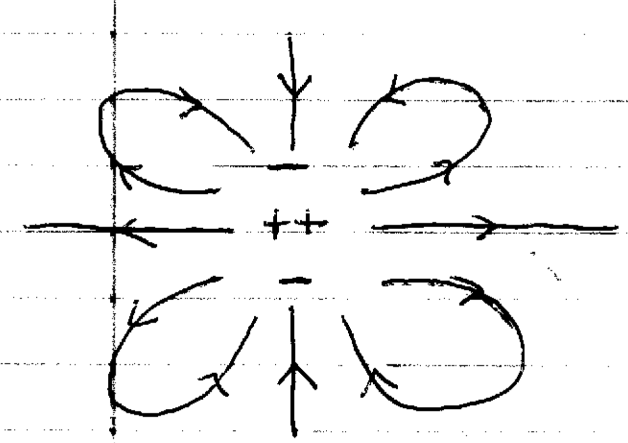
$$\mathbf{E} = -\nabla\left(\frac{1}{4\pi\epsilon_0}\,\frac{\mathbf{p}\cdot\mathbf{x}}{x^3}\right) = \frac{1}{4\pi\epsilon_0}\left(\frac{\mathbf{p}}{x^3} + \frac{3\,\mathbf{p}\cdot\mathbf{x}}{x^5}\,\mathbf{x}\right)$$

$$\nabla(\mathbf{p}\cdot\mathbf{x}) = \hat{i}\,\frac{\partial}{\partial x_i}\, p_j x_j = \hat{i}\, p_j \delta_{ij} = \hat{i}\, p_i = \mathbf{p}$$

$$\nabla\frac{1}{x^3} = \nabla\frac{1}{(\mathbf{x}\cdot\mathbf{x})^{3/2}} = -\frac{3}{2}\,\frac{1}{(\mathbf{x}\cdot\mathbf{x})^{5/2}}\, 2\mathbf{x}$$

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\left[\,3\,\mathbf{p}\cdot\hat{r}\,\hat{r} - \mathbf{p}\,\right]\frac{1}{r^3}$$

Quadrupole electric field

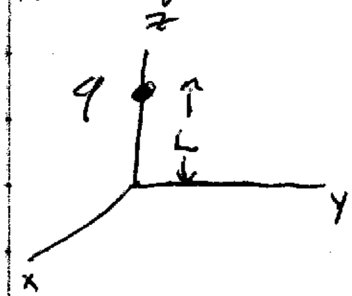


Dependence on the choice of the origin:

The multipole moments generally depend on the choice of the origin.

$\Rightarrow$ the lowest order non-zero moment is independent of the choice of the origin

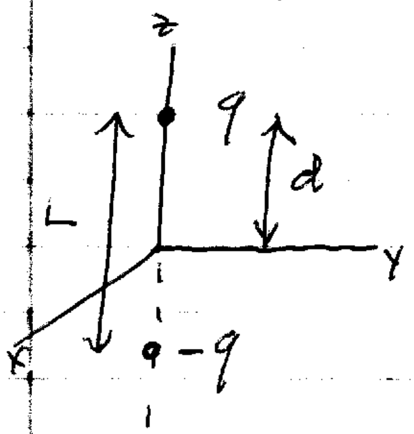
<u>example</u>: point charge



total charge : q origin indep.

dipole moment : $p = qL$ origin dep.

<u>example</u> : dipole



total charge : $q = 0$

dipole moment : $p_z = qd - q(d-L)$
$= qL$

$\Rightarrow$ indep. of origin

<u>Multipole moments from the Y_ℓ^m's.</u>

$$\frac{1}{|\underset{\wedge}{x} - \underset{\wedge}{x}'|} = 4\pi \sum_{\ell,m} \frac{Y_\ell^{m*}(\theta',\varphi')\, Y_\ell^m(\theta,\varphi)}{2\ell+1} \frac{x'^{\ell}}{x^{\ell+1}}$$

$$\Phi(\underline{x}) = \frac{1}{\epsilon_0} \sum_{l,m} \frac{1}{2l+1} q_{lm} Y_l^m(\theta,\varphi) \frac{1}{x^{l+1}}$$

$$q_{lm} = \int d\underline{x}' \, x'^l \, Y_l^{m*}(\theta',\varphi') \rho(\underline{x}')$$

Potential energy

Consider an external potential $\Phi(\underline{x})$ produced by some remote distribution of charges. Bring a fixed distribution of charge $\rho(\underline{x})$ into $\Phi(\underline{x})$.

$\Rightarrow$ What is the energy associated with the introduction of ρ into Φ?

$\Rightarrow$ Do not include the energy required to assemble $\rho(\underline{x})$.

$$W = \int d\underline{x} \, \rho(\underline{x}) \, \Phi(\underline{x})$$

Assume $\Phi(\underline{x})$ varies only weakly over the distribution of $\rho(\underline{x})$

$\Rightarrow$ expand $\Phi(\underline{x})$ in a Taylor series around some origin in $\rho(\underline{x})$

$$\Phi(\underset{\sim}{x}) = \Phi(0) + \overbrace{\underset{\sim}{x}\cdot\nabla\Phi}^{-\underset{\sim}{E}\cdot\underset{\sim}{x}} + \overbrace{\frac{1}{2}\, x_i x_j \frac{\partial^2 \Phi}{\partial x_i \partial x_j}}^{-\frac{1}{2} x_i x_j \frac{\partial}{\partial x_j} E_i}$$

$$+ \cdots$$

Since Φ is an external potential

$$\nabla^2 \Phi = 0 \Rightarrow \nabla\cdot\underset{\sim}{E} = 0$$

$$\delta_{ij} \frac{\partial}{\partial x_j} E_i = 0$$

$$\Phi(\underset{\sim}{x}) = \Phi(0) - \underset{\sim}{E}(0)\cdot\underset{\sim}{x} - \frac{1}{6}\left(3 x_i x_j - \delta_{ij} x^2\right)\frac{\partial}{\partial x_j} E_i$$

$$+ \cdots$$

<u>Components of Φ</u>

monopole component : $\Phi(0)$

dipole component : $-\underset{\sim}{E}(0)$

quadrupole component : $\frac{\partial}{\partial x_j} E_i$

$$W = q\,\Phi(0) - \underset{\sim}{p}\cdot\underset{\sim}{E}(0) - \frac{1}{6}\sum_{i,j} Q_{ij} \frac{\partial}{\partial x_j} E_i$$

$$+ \cdots$$

The energy stored W is a sum of the products of the multipole components of ρ with the corresponding multipole component of Φ.

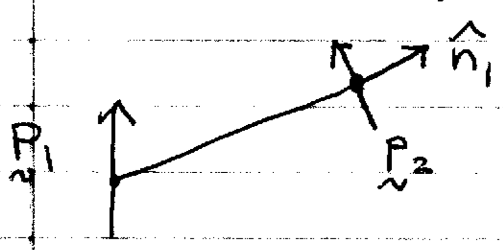
$\Rightarrow$ e.g. monopole component of ρ times monopole component of Φ is

$$q\,\Phi(0)$$

$\Rightarrow$ like components interact

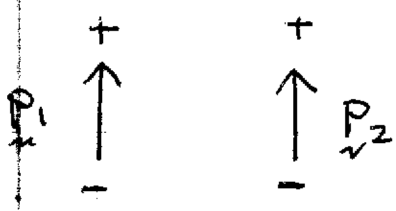
<u>example</u> dipole - dipole interaction

How does a dipole respond to a fixed dipole?



$$W = -\underset{\sim}{P}_2 \cdot \underset{\sim}{E}_1 = -\underset{\sim}{P}_2 \cdot \left[\frac{3\,\hat{n}_1\, \underset{\sim}{P}_1 \cdot \hat{n}_1 - \underset{\sim}{P}_1}{|\underset{\sim}{x}_1 - \underset{\sim}{x}_2|^3} \right]$$

$$= \left(\underset{\sim}{P}_1 \cdot \underset{\sim}{P}_2 - 3\, \hat{n}_1 \cdot \underset{\sim}{P}_1\, \hat{n}_1 \cdot \underset{\sim}{P}_2 \right) \frac{1}{|\underset{\sim}{x}_1 - \underset{\sim}{x}_2|^3}$$



$$w \sim \frac{p_1 p_2}{x_{12}^3} \quad \text{repulsive}$$

$\Rightarrow$ like changes closest together

$$w \sim -\frac{p_1 p_2}{x_{12}^3} \quad \text{attractive}$$

$\Rightarrow$ opposite changes closest together

How do dipoles want to orient and what are the net forces?