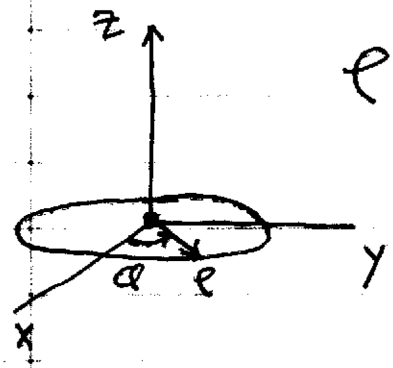
Basis functions in Cylindrical coordinates



ρ, φ, z

First consider a 2-D system: $\frac{\partial}{\partial z} = 0$

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \varphi^2}$$

Laplace's eqn of basis function h,

$$\left(\frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \varphi^2}\right) h(\rho,\varphi) = 0$$

Let $h = R(\rho)\,\Phi(\varphi)$

$$\frac{1}{\rho^2} h\Big[\underbrace{\frac{1}{R}\rho\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho}R(\rho)}_{\nu^2} + \underbrace{\frac{1}{\Phi}\frac{\partial^2}{\partial \varphi^2}\Phi}_{-\nu^2}\Big] = 0$$

$$\frac{\partial^2}{\partial \varphi^2}\Phi + \nu^2\Phi = 0 \implies \Phi \sim e^{\pm i\nu\varphi}$$

$\Rightarrow$ BCs in φ control ν

$\Rightarrow$ for $\varphi \in (0, 2\pi)$, ν is an integer

$$\rho \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} R - \nu^2 R = 0$$

$\Rightarrow$ Euler eqn $\Rightarrow R \sim \rho^{\gamma}$

$$(\gamma^2 - \nu^2)\rho^{\gamma} = 0$$

$$\gamma = \pm \nu \Rightarrow R \sim \rho^{\pm \nu}$$

For $\nu = 0 \Rightarrow R \sim \rho^0, \ln(\rho)$

$\Rightarrow$ oscillatory in φ and non-oscillatory in ρ

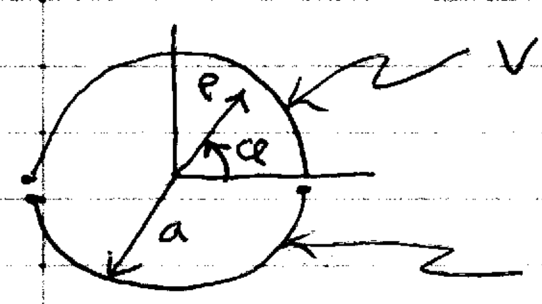
$\Rightarrow$ can choose solutions oscillatory in ρ and exponential in φ.

General solution:

$$\Phi = \sum_{\nu} \left(a_\nu e^{i\nu\varphi} + b_\nu e^{-i\nu\varphi} \right)\left(d_\nu \rho^{\nu} + e_\nu \rho^{-\nu} \right)$$

$\Rightarrow$ BCs in φ control ν.

Example Potential inside cylindrical shell



$$\Phi = V \text{ for } \varphi \in (0, \pi)$$
$$= -V \text{ for } \varphi \in (\pi, 2\pi)$$

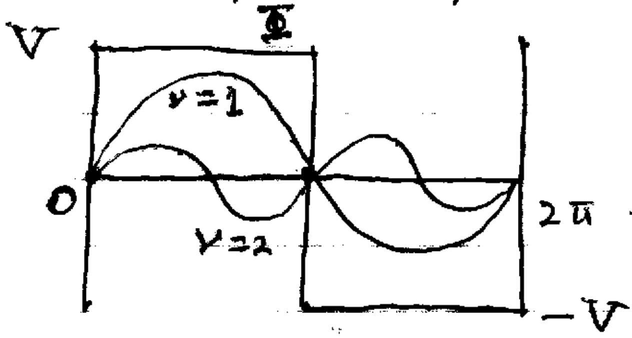
$\Rightarrow$ ν an integer since solution must be periodic over 2π

$\Rightarrow$ $\Phi \sim \sin(\nu\varphi)$, $\cos(\nu\varphi)$

$\Rightarrow$ Φ is an odd function across $\varphi = 0$

$\Rightarrow$ $\Phi \sim \sin\nu\varphi$

Even symmetry around $\varphi = \pi/2$



$\Rightarrow$ only ν odd contribute

$\Rightarrow$ no contribution from $\nu = 0$

Dependence on ρ $\Rightarrow$ discard $\rho^{-\nu}$

$$\Phi = \sum_{\substack{\nu=1 \\ \nu \text{ odd}}}^{\infty} b_\nu \left(\frac{\rho}{a}\right)^\nu \sin\nu\varphi$$

BC at $\rho = a$

$$\Phi(a,\varphi) = \sum_\nu b_\nu \sin\nu\varphi$$

$$\underbrace{\int_{-\pi}^{\pi} d\varphi \, \sin\nu'\varphi \, \Phi(a,\varphi)} = \frac{1}{2}(2\pi)\, b_{\nu'}$$

$$\underbrace{V\left[\int_0^{\pi} d\varphi \sin\nu'\varphi - \int_{-\pi}^{0} d\varphi \sin\nu'\varphi\right]}$$

$$2V \underbrace{\int_0^{\pi} d\varphi \sin\nu'\varphi} \quad \text{since } \sin\nu'\varphi \text{ odd in } \varphi$$

$$-\frac{\cos\nu'\varphi}{\nu'}\Big|_0^{\pi} = -\left(\frac{\cos\nu'\pi - 1}{\nu'}\right)$$

$$= \frac{2}{\nu} \text{ for } \nu \text{ odd}$$

$$\frac{4V}{\pi\nu} = b_\nu$$

$$\Phi = \frac{4V}{\pi} \sum_{\nu\,\text{odd}} \frac{1}{\nu}\left(\frac{\rho}{a}\right)^{\nu} \sin\nu\varphi$$

<u>This series can be summed</u>

$$\Phi = \frac{4V}{\pi}\, \mathrm{Im} \sum_{\nu} \frac{1}{\nu}\left(\frac{\rho}{a} e^{i\varphi}\right)^{\nu}$$

Let $z = \frac{\rho}{a} e^{i\varphi}$

Let $S = \sum_{\nu} \frac{1}{\nu} z^{\nu}$

$$\frac{dS}{dz} = \sum_{\nu} z^{\nu-1} = 1 + z^2 + z^4 \ldots$$

$$= \frac{1}{1-z^2}$$

$$\frac{d\mathcal{S}}{dz} = \left(\frac{1}{1-z} + \frac{1}{1+z}\right)\frac{1}{2}$$

$$\mathcal{S} = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)$$

$$\frac{1+z}{1-z} = \frac{(1+z)(1-z^*)}{|1-z|^2} = \frac{1-|z|^2+2i\,\mathrm{Im}\,z}{|1-z|^2}$$

$$= \left|\frac{1+z}{1-z}\right| e^{i\Theta} = |\ \ |(\cos(\Theta)+i\sin(\Theta))$$

$$\Theta = \tan^{-1}\left[\frac{2\,\mathrm{Im}(z)}{1-|z|^2}\right] = \tan^{-1}\left[\frac{2\frac{\rho}{a}\sin\varphi}{1-\rho^2/a^2}\right]$$

$$\Phi = \frac{4V}{\pi}\,\mathrm{Im}\,\mathcal{S} = \frac{2V}{\pi}\,\Theta$$

$$\Phi = \frac{2V}{\pi}\tan^{-1}\left[\frac{2a\rho\sin\varphi}{a^2-\rho^2}\right]$$

see Jackson 2.13

Basis functions in 3-D cylindrical coordinates

In a 3D cylinder

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2}$$

Basis functions

$$h(\rho,\varphi,z) = R(\rho)\,Q(\varphi)\,Z(z)$$

$$0 = \nabla^2 h = h\left[\frac{1}{R}\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}R + \frac{1}{\rho^2 Q}\frac{\partial^2}{\partial\varphi^2}Q + \frac{1}{Z}\frac{\partial^2}{\partial z^2}Z\right]$$

Choose oscillatory in ρ, φ and ~~exponti~~ exponential in z.

$$\frac{\partial^2}{\partial\varphi^2}Q + \nu^2 Q = 0, \quad Q \sim e^{\pm i\nu\varphi}$$

When $\varphi \in (0, 2\pi) \Rightarrow \nu = \text{integer}$

$$\frac{d^2}{dz^2}Z - k^2 Z = 0 \Rightarrow Z \sim e^{\pm kz}$$

$$0 = h\left[\frac{1}{R}\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}R - \frac{\nu^2}{\rho^2} + k^2\right] = 0$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}R + \left(k^2 - \frac{\nu^2}{\rho^2}\right)R = 0$$

$\Rightarrow$ Bessel's eqn.

$$\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho}R - \frac{\nu^2}{\rho}R + k^2\rho R = 0$$

Compare with Stürm-Liouville form

$$\frac{\partial}{\partial x}P(x)\frac{\partial}{\partial x}u_n + q(x)u_n + w(x)\lambda_n u_n = 0$$

$$\Rightarrow P \to \rho,\ q \to -\frac{\nu^2}{\rho},\ \lambda = k^2,\ w = \rho$$

weight function : ρ
eigenvalue : k^2

boundary condition :

$$\rho R(\rho) R'(\rho) \Big|_0^a = 0$$

$\Rightarrow$ basis functions must be bounded at $\rho = 0$

$\Rightarrow$ BCs satisfied as long as R, R' bounded at $\rho = 0$

$\Rightarrow$ require $R(a)$ or $R'(a) = 0$

Behavior of solutions near $\rho = 0$

$$\rho^2 \frac{d^2}{d\rho^2} R + \rho \frac{d}{d\rho} R + (\cancel{k^2\rho^2} - \nu^2) R = 0$$

small

$$\rho^2 \frac{d^2}{d\rho^2} R + \rho \frac{d}{d\rho} R - \nu^2 R = 0$$

$\Rightarrow$ Euler eqn $\Rightarrow$ powerlaw solution

$$R \sim \rho^\gamma$$

$$[\gamma(\gamma-1) + \gamma - \nu^2]\rho^\gamma = 0$$

$$\gamma = \pm\nu \Rightarrow R \sim \rho^{\pm\nu}$$

$\Rightarrow$ one bounded and one divergent solution

Standard solutions of Bessel's Eqn:

Let $x = k\rho$, $x^2 \frac{d^2}{dx^2} R_\nu + x \frac{d}{dx} R_\nu + (x^2 - \nu^2) R_\nu = 0$

Bessel function of first kind

$$J_\nu(x), \quad J_{-\nu}(x)$$

For $x \ll 1 \Rightarrow J_\nu \simeq \frac{(x/2)^\nu}{\Gamma(\nu+1)}$

$$J_{-\nu} \simeq \frac{(x/2)^{-\nu}}{\Gamma(1-\nu)}$$

Note that J_ν and $J_{-\nu}$ are degenerate for ν an integer.

Second solution given by Neumann function

$$N_\nu(x) = \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin \nu\pi}$$

$x \ll 1$:

$$N_\nu = -\frac{\Gamma(\nu)}{\pi}\left(\frac{2}{x}\right)^\nu, \quad \nu \neq 0$$

$$= \frac{2}{\pi}\left[\ln\left(\frac{x}{2}\right) + .5772\ldots\right], \quad \nu = 0$$

$\Rightarrow$ $N_\nu(x)$ diverges at $x = 0$ for all ν

$\Rightarrow$ ~~Discard $N_\nu(x)$ when $\ell = 0$~~ is included

$x \gg 1$:

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$$

$$N_\nu(x) = \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$$

$\Rightarrow$ WKB solutions

Hankel functions:

$$H_\nu^{(1)} = J_\nu + i N_\nu$$

$$H_\nu^{(2)} = J_\nu - i N_\nu$$

For large $x \sim e^{\pm ix}$

$\Rightarrow$ important for wave problems

Modified Bessel functions

If behavior in z is oscillatory,

$$Z \sim e^{\pm ikz}$$

The radial dependence must be non-oscillatory

$$x^2 \frac{d^2}{dx^2} R_\nu + x \frac{d}{dx} R_\nu \cancel{+(x^2-\nu^2)} - (x^2+\nu^2) R_\nu = 0$$

Solutions:

$$I_\nu(x) = i^{-\nu} J_\nu(ix)$$

$$K_\nu(x) = \frac{\pi}{2} i^{\nu+1} H_\nu^{(1)}(ix)$$

$x \ll 1$: $\quad I_\nu \sim x^\nu, \; K_\nu \to \infty$

$x \gg 1$:

$$\left.\begin{aligned} I_\nu &\sim \frac{e^{x}}{\sqrt{x}} \\ K_\nu &\sim \frac{e^{-x}}{\sqrt{x}} \end{aligned}\right\} \text{exponential in } \rho$$

The Bessel function $J_\nu(k\rho)$ satisfies Stürm-Liouville BC.

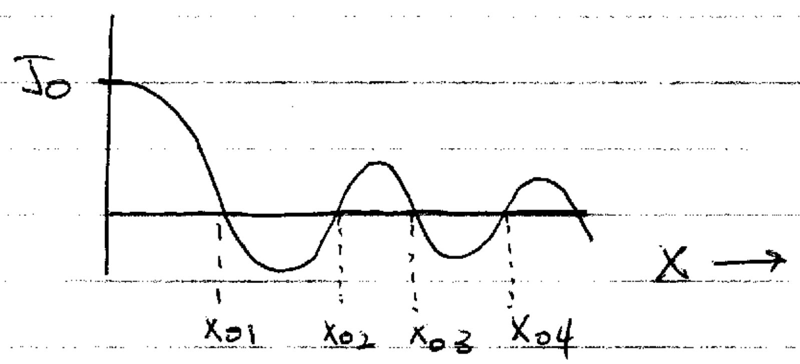
$$\rho\, J_\nu(k\rho)\, J_\nu'(k\rho) \Big|_0^a = 0$$

at $\rho = 0 \Rightarrow J_\nu$ bounded.

Choose $J_\nu(ka) = 0$

$$k_{\nu n}\, a = x_{\nu n}, \quad n = 1, 2 \ldots$$

$x_{\nu n}$ is the nth zero of $J_\nu(x)$



Can also choose $J_\nu'(k_{\nu n} a) = 0$

$$k_{\nu n} = \frac{y_{\nu n}}{a}$$

$y_{\nu n}$ = nth zero of the derivative of $J_\nu(x)$

weight function

$$\int_0^a d\rho\, \rho\, J_\nu\left(x_{\nu n}\frac{\rho}{a}\right) J_\nu\left(x_{\nu n'}\frac{\rho}{a}\right) = \delta_{nn'}\, \frac{a^2}{2} \times J_{\nu+1}^2(x_{\nu n})$$

The $J_\nu\left(x_{\nu n}\frac{\rho}{a}\right)$ with $n = 1, 2 \ldots \infty$ form a complete over $x \in (0, a)$

$$g(\rho) = \sum_{n=1}^{\infty} a_{n\nu} J_\nu\left(x_{\nu n}\frac{\rho}{a}\right)$$

$\Rightarrow$ Fourier Bessel series

$\Rightarrow$ Calculate $a_{n\nu}$ using orthogonality

$$a_{n\nu} = \frac{\int_0^a d\rho'\, \rho'\, g(\rho')\, J_\nu\left(x_{\nu n}\frac{\rho'}{a}\right)}{\frac{a^2}{2} J_{\nu+1}^2(x_{\nu n})}$$

Insert into $g(\rho)$ above

$$g(\rho) = \int_0^a d\rho'\, g(\rho') \underbrace{\sum_{n=1}^{\infty} \rho' \frac{J_\nu\left(x_{\nu n}\frac{\rho'}{a}\right) J_\nu\left(x_{\nu n}\frac{\rho}{a}\right)}{\frac{a^2}{2} J_{\nu+1}^2(x_{\nu n})}}_{\delta(\rho - \rho')}$$

$\Rightarrow$ completeness relation

How do the basis functions change for infinite systems?

$$\Longrightarrow a \to \infty$$

$$k_n = \frac{x_{\nu n}}{a} \Rightarrow k_n a = \cancel{\;} x_{\nu n}$$

$\Longrightarrow$ dominated by n large

$$J_\nu \sim \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$$

$\Rightarrow$ zeros of J_ν separated by π

$$k_{n+1} - k_n = \Delta k = \frac{x_{\nu\, n+1} - x_{\nu n}}{a} = \frac{\pi}{a} \longrightarrow 0$$

Sum over n becomes an integral over k

$$\sum_n \Rightarrow \sum_n dn \Rightarrow \int dk \frac{a}{\pi}$$

Fourier Bessel series $\Longrightarrow$

$$g(\rho) = \int_0^\infty dk \; G_\nu(k) J_\nu(k\rho)$$

Completeness:

$$\delta(\rho - \rho') = \rho' \int_0^\infty dk \frac{a}{\pi} \frac{J_\nu(k\rho') J_\nu(k\rho)}{\frac{a^2}{2} J_{\nu+1}^2(x_{\nu n})}$$

$$J_{\nu+1}(x_{\nu n}) = \sqrt{\frac{2}{\pi x_{\nu n}}} \cos\left[x_{\nu n} - (\nu+1)\frac{\pi}{2} - \frac{\pi}{4}\right]$$

Since $J_\nu(X_{\nu n}) = 0$,

$$\cos\left(X_{\nu n} - \nu\frac{\pi}{2} - \frac{\pi}{4}\right) = 0$$

$$\Rightarrow X_{\nu n} - \nu\frac{\pi}{2} - \frac{\pi}{4} = (m+\tfrac{1}{2})\pi \quad \text{with } m \text{ an intege}$$

Therefore

$$J_{\nu+1}(X_{\nu n}) = \sqrt{\frac{2}{\pi X_{\nu n}}}\cos\Big[\underbrace{(m+\tfrac{1}{2})\pi - \frac{\pi}{2}}_{m\pi}\Big]$$

$$= \sqrt{\frac{2}{\pi X_{\nu n}}}\,(\pm 1)$$

$$\delta(\rho-\rho') = \rho'\int_0^\infty dk\,\frac{a}{\pi}\,\frac{J_\nu(k\rho')J_\nu(k\rho)}{\frac{a^2}{2}\,\frac{2}{\pi}\,\frac{1}{X_{\nu n}}}$$

But $k = \dfrac{X_{\nu n}}{a}$

$$\delta(\rho-\rho') = \rho'\int_0^\infty dk\,k\,J_\nu(k\rho')J_\nu(k\rho)$$

Can also interchange k with ρ,

$$\delta(k-k') = k'\int_0^\infty d\rho\,\rho\,J_\nu(k'\rho)J_\nu(k\rho)$$

Laplace's eqn in cylindrical coordinates

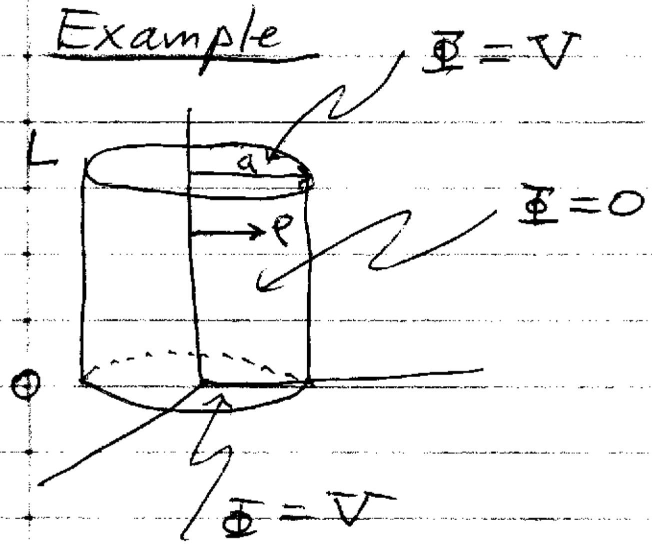
choose $\varphi \in (0, 2\pi)$

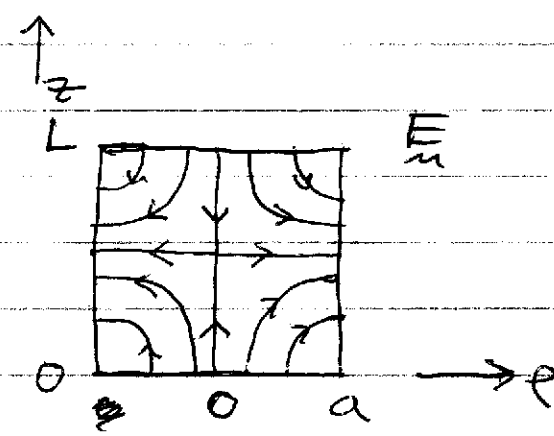
$$\nabla^2 \Phi = 0$$

$$\Phi(\rho,\varphi,z) = \sum_{\nu,n} \left(a \sinh(k_{\nu n} z) + b \cosh(k_{\nu n} z) \right) \otimes \left(c J_\nu(k_{\nu n}\rho) + d N_\nu(k_{\nu n}\rho) \right) \otimes \left(e \sin\nu\varphi + f \cos\nu\varphi \right)$$

$\Rightarrow$ $k_{\nu n}$ determined by BC in ρ.

Example





$$\Phi = \sum_n C_n \cosh\left[k_{0n}\left(z - \tfrac{L}{2}\right)\right] J_0(k_{0n}\rho)$$

where $\nu = 0$ because of azimuthal symmetry, $N_0(k_{0n}\rho)$ discarded since solution bounded at $\rho = 0$, and invoked symmetry of the solution around $z = \frac{L}{2}$.

Since $\Phi(\rho=a, z) = 0$, take

$$J_0(k_{0n} a) = 0$$

$$\Rightarrow k_{0n} = \frac{x_{0n}}{a}, \text{ with } x_{0n} \text{ zeros of } J_0$$

To calculate C_n, use BC $\Phi = V$ at $z = L$,

$$V = \sum_{n=1}^{\infty} C_n \cosh\left(k_{0n}\tfrac{L}{2}\right) J_0(k_{0n}\rho)$$

Multiply by $\rho J_0(k_{0n'}\rho)$ and integrate over ρ.

$$\Rightarrow C_n = V \frac{2}{\cancel{a^2}} \frac{\overbrace{\int_0^a \frac{d\rho\, \rho}{a^2} J_0\left(x_{0n}\frac{\rho}{a}\right)}^{I}}{\cosh\left(k_{0n}\frac{L}{2}\right) J_1^2(x_{0n})}$$

$x \equiv \frac{\rho}{a} x_{0n}$

$$= 2V\ I \frac{1}{\cosh\left(k_{0n}\frac{L}{2}\right) J_1^2(x_{0n})}$$

$$I = \frac{1}{x_{0n}^2} \int_0^{x_{0n}} dx\, x\, J_0(x)$$

Use recursion formula to carry out the integral

$$\frac{d}{dx} x^p J_p = x^p J_{p-1}(x)$$

$$\frac{d}{dx} x J_1(x) = x J_0(x)$$

$$I = \int_0^{x_{0n}} dx \frac{\frac{d}{dx}\left[x J_1(x)\right]}{x_{0n}^2} = \left.\frac{x J_1(x)}{x_{0n}^2}\right|_0^{x_{0n}}$$

$$= \frac{J_1(x_{0n})}{x_{0n}}.$$

$$C_n = 2V \frac{1}{x_{0n}} \frac{1}{\cosh\left(\frac{x_{0n}L}{2a}\right)} \frac{1}{J_1(x_{0n})}$$

$$\Phi = 2V \sum_{n=1}^{\infty} \frac{\cosh \frac{x_{0n}}{a}\left(z - \frac{L}{2}\right)}{x_{0n} \cosh\left(\frac{x_{0n}L}{2a}\right)} \cdot \frac{J_0\left(\frac{x_{0n}\rho}{a}\right)}{J_1(x_{0n})}$$

How will the solution behave in the limit $L \gg a$

$\Rightarrow$ top and bottom potentials should decouple

$\Rightarrow$ want to show this

$$\cosh\left(\frac{x_{0n}L}{2a}\right) \simeq \frac{1}{2} e^{\frac{x_{0n}L}{2a}} \gg 1$$

Near $z = L$,

$$\Phi = 2V \sum_{n=1}^{\infty} \underbrace{\frac{e^{\frac{x_{0n}}{a}\left(z - \frac{L}{2}\right)}}{e^{\frac{x_{0n}L}{2a}}}}_{e^{\frac{x_{0n}}{a}(z-L)}} \frac{J_0\left(\frac{x_{0n}\rho}{a}\right)}{J_1(x_{0n})}$$

Solution of Φ decays to small value for

$$\frac{(L-z)\,x_{0n}}{a} \sim 1$$

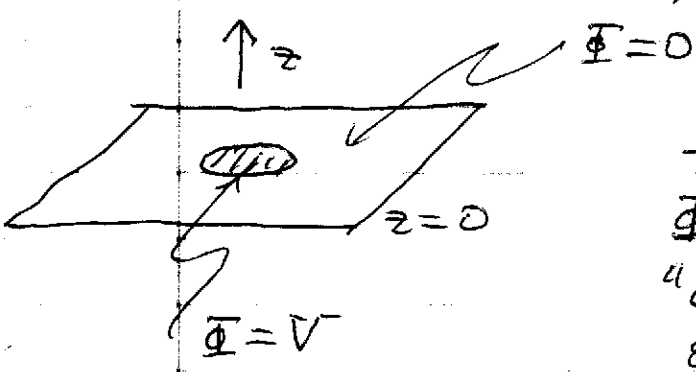
$\Rightarrow$ becomes small over a distance
$L - z \sim a << L$

Note also that for $\frac{L-z}{a} >> 1$, the dominant contribution comes from $n = 1$ since x_{01} is smaller than $x_{02}, x_{03} \ldots$

So

$$\Phi \simeq 2V\, e^{\frac{x_{01}(z-L)}{a}} \frac{J_0\left(\frac{x_{01}\rho}{a}\right)}{x_{01} J_1(x_{01})}.$$

<u>Example</u> Open system with $\rho \to \infty$



Solve for Φ above the plane at $z=0$ where $\Phi = V$ in a disk of radius "a" and zero for $\rho > a$ on the plane.

Choose oscillatory functions in ρ so can match Φ at $z=0$.

$$\Phi = \int_0^\infty dk\, e^{-kz} J_0(k\rho)\, g(k)$$

where

1) Only $\nu = 0$ contributes
$\Longrightarrow$ symmetry in φ
2) Linear combination of $\cosh(kz)$ and $\sinh(kz)$ so that $\Phi \to 0$ as $z \to \infty$, $\sim e^{-kz}$
3) $N_0(k\rho)$ discarded since Φ bounded at $\rho = 0$.

$$\Phi(z=0, \rho) = \int_0^\infty dk\, g(k)\, J_0(k\rho)$$

Want to eliminate the integral over k to solve for $g(k)$.

$\Longrightarrow$ Multiply by $\rho J_0(k'\rho)$ and integrate over ρ

$$\underbrace{\underbrace{V \int_0^a d\rho\, \rho\, J_0(k'\rho)}_{k'\rho \equiv x}}_{V \frac{1}{k'^2} \int_0^{k'a} dx\, x\, J_0(x)} = \int_0^\infty dk\, g(k) \underbrace{\int_0^\infty d\rho\, \rho\, J_0(k\rho) J_0(k'\rho)}_{\frac{1}{k}\delta(k-k')}$$

$$V \frac{1}{k'^2} \int_0^{k'a} dx\, x\, J_0(x) = g(k') \frac{1}{k'}$$

$$x J_0(x) = \frac{d}{dx} x J_1(x)$$

$$g(k') = \frac{V}{k'} \int_0^{ka} dx \, \frac{d}{dx} x J_1$$

$$= \frac{V}{k'} \, k'a \, J_1(k'a) = Va \, J_1(k'a)$$

$$\Phi = V a \int_0^{\infty} dk \, e^{-kz} J_0(k\rho) J_1(ka)$$

Behavior for large z at $\rho = 0$? Take $z \gg a$.

$$\Phi \simeq V a \int_0^{\infty} dk \, e^{-kz} J_1(ka)$$

Because z is large the dominant contribution to the integral over k must come from

$$k \lesssim \frac{1}{z} \quad \text{or} \quad ka \lesssim \frac{a}{z} \ll 1$$

Can expand $J_1(ka)$ for small argument.

$$J_1(ka) \simeq \frac{ka}{2} \frac{1}{\Gamma(2)} = \frac{ka}{2}$$

$$\Phi \simeq V a \, \frac{a}{2} \int_0^{\infty} dk \, e^{-kz} k$$

$$= V \frac{a^2}{2} \frac{1}{z^2} \int_0^{\infty} ds \, s e^{-s}$$

$$\sim V \frac{a^2}{z^2} \qquad \text{Why } \frac{1}{z^2}?$$

Infinite medium Green's function in cylindrical coordinates

Consider a delta function source at $\underset{\sim}{x}'$,

$$\delta(\underset{\sim}{x}-\underset{\sim}{x}') = \frac{1}{\rho}\,\delta(\rho-\rho')\,\delta(\varphi-\varphi')\,\delta(z-z')$$

$$1 = \int d\underset{\sim}{x}\;\delta(\underset{\sim}{x}-\underset{\sim}{x}') = \int_0^{\infty} d\rho \int_0^{2\pi} d\varphi \int_{-\infty}^{\infty} dz'\,\rho\,\delta(\underset{\sim}{x}-\underset{\sim}{x}')$$

$$\nabla^2 G(\underset{\sim}{x},\underset{\sim}{x}') = -4\pi\,\delta(\underset{\sim}{x}-\underset{\sim}{x}') = -\frac{4\pi}{\rho}\,\delta(\rho-\rho')\,\delta(\varphi-\varphi') \otimes \delta(z-z')$$

$\Rightarrow$ choose functions periodic in z, φ

$\Rightarrow$ even function of $z-z'$

$\Rightarrow$ only a function of $\varphi-\varphi'$

$$G = \sum_{m=-\infty}^{\infty} \int_0^{\infty} dk\; e^{im(\varphi-\varphi')} \cos[k(z-z')]\, g_{mk}(\rho,\rho')$$

$\Rightarrow$ insert into Poisson's eqn.

$\Rightarrow$ eliminate $\varphi-\varphi'$, $z-z'$ dependence using orthogonality.

$\Rightarrow$ yields equation in ρ for g_{mk}

$\Rightarrow$ modified Bessel eqn with source

$$G(\underset{\sim}{x}, \underset{\sim}{x}') = \frac{1}{|\underset{\sim}{x} - \underset{\sim}{x}'|} = \frac{2}{\pi} \sum_{m=-\infty}^{\infty} e^{im(\varphi - \varphi')}$$

$$\otimes \int_0^\infty dk \cos[k(z-z')] \, I_m(k\rho_<) K_m(k\rho_>)$$

with $\rho_<$ = smaller of ρ, ρ'
$\rho_>$ = larger of ρ, ρ'

I_m, K_m modified Bessel functions

$K_m(k\rho) \longrightarrow 0$ as $\rho \longrightarrow \infty$
$I_m(k\rho)$ bounded as $\rho \longrightarrow 0$