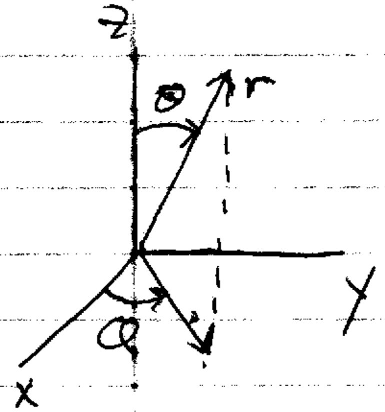
Basis functions in spherical coordinates

$\Rightarrow$ Laplace's eqn

$$\nabla^2 \varphi = 0$$

Consider basis functions



$$h(r,\theta,\varphi) = P(\theta)\,\Phi(\varphi)\,R(r)$$

$\Rightarrow$ separable

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\sin\theta\frac{\partial}{\partial \theta}$$

$$+ \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \varphi^2}$$

$$\nabla^2 h = \frac{h}{r^2}\left[\frac{1}{R}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} R + \frac{1}{P\sin\theta}\frac{\partial}{\partial \theta}\sin\theta\frac{\partial}{\partial \theta} P \right.$$

$$\left. + \frac{1}{\sin^2\theta}\,\frac{1}{\Phi}\frac{\partial^2}{\partial \varphi^2}\Phi\right] = 0$$

$$\Rightarrow \frac{1}{\Phi}\frac{\partial^2}{\partial \varphi^2}\Phi = \text{const} \equiv -m^2$$

$\Rightarrow$ since other terms in [ ] indep. of φ.

$$\frac{\partial^2}{\partial \varphi^2}\Phi + m^2\Phi = 0 \quad \Rightarrow \Phi \sim e^{\pm i m \varphi}$$

In a system where $\varphi \in (0, 2\pi)$, m is an integer

$\Rightarrow$ periodic over 2π

Remaining terms in the bracket yield

$$\underbrace{\frac{1}{R}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}R}_{\substack{\text{depends only}\\ \text{on } r \\ \ell(\ell+1)}} + \underbrace{\frac{1}{\sin\theta}\frac{1}{P}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}P - \frac{m^2}{\sin^2\theta}}_{\substack{\text{depends only on } \theta \\ -\ell(\ell+1)}} = 0$$

$$\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}R_\ell - \ell(\ell+1)R_\ell = 0$$

$\Rightarrow$ Euler eqn

$\Rightarrow$ powerlaw solutions

$$R \sim r^\gamma$$

$$[\gamma(\gamma+1) - \ell(\ell+1)]r^\gamma = 0$$

$$\gamma = \ell, -\ell-1$$

$$\Rightarrow R \sim r^\ell, r^{-\ell-1}$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{d}{d\theta}P_\ell^m + \left[\ell(\ell+1) - \frac{m^2}{\sin^2\theta}\right]P_\ell^m = 0$$

$$x = \cos\theta$$

$$dx = -\sin\theta\, d\theta$$

$$\frac{d}{dx}(1-x^2)\frac{d}{dx}P_\ell^m + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right]P_\ell^m = 0$$

Associated Legendre eqn

solutions P_ℓ^m, Q_ℓ^m

Behavior near $x=1 \Rightarrow$ Let $t=1-x$

$$1-x^2=(1+x)(1-x) \overset{v}{=} 2t, \quad dt=-dx$$

$$\left[\frac{d}{dt}2t\frac{d}{dt}+\ell(\ell+1)-\frac{m^2}{2t}\right]P=0$$

small

$$\left[2t\frac{d}{dt}t\frac{d}{dt}+\cancel{t(\ell+1)\ell}-\frac{m^2}{2}\right]P=0$$

$\Rightarrow$ Euler eqn $\Rightarrow$ powerlaw solution

$$P \sim t^\gamma$$

$$\left(2\gamma^2-\frac{m^2}{2}\right)t^\gamma=0$$

$$\gamma=\pm\frac{m}{2} \Rightarrow P \sim (1-x)^{\pm\frac{m}{2}}$$

$\Rightarrow$ one singular and one non-singular solution

$\Rightarrow$ series solutions in powers of

$$P \sim \sum_{n=0}^{\infty} a_n x^n$$

diverge at $x=\pm 1$ unless the series truncates.

$\Rightarrow$ requires ℓ be an integer

P_ℓ^m bounded at $x = \pm 1$ for ℓ an integer

Q_ℓ^m diverges $\Rightarrow$ discard

For $m=0$, near $x=1$

$$t \frac{d}{dt} t \frac{d}{dt} P_\ell^0 = 0$$

$$t \frac{d}{dt} P = \text{const} \Rightarrow P \sim \text{const.} + \ln(t)$$

$\Rightarrow$ one convergent and one divergent solution at $x = \pm 1$.
$\theta = 0, \pi$

For $m=0$, have Legendre's eqn

$$\frac{d}{dx}(1-x^2)\frac{d}{dx}P_\ell + \ell(\ell+1)P_\ell = 0$$

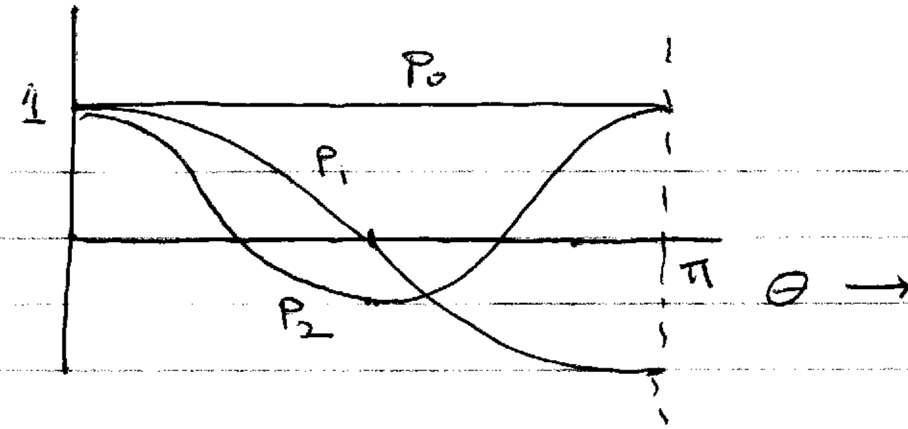
Again, series truncates for ℓ an integer

$\Rightarrow$ Legendre Polynomials bounded at $x = \pm 1$

Rodriquez formula

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\frac{d^\ell}{dx^\ell}(x^2-1)^\ell$$

$P_0 = 1$, $P_1 = x$, $P_2 = \frac{1}{2}(3x^2-1)$, $P_\ell \sim x^\ell$



$\Rightarrow$ oscillatory

$\Rightarrow$ form a complete set over $(-1, 1)$

Orthogonality

$$\int_{-1}^{1} dx\, P_\ell(x) P_{\ell'}(x) = \frac{2}{2\ell+1} \delta_{\ell\ell'}$$

Recursion relations

$$(\ell+1) P_{\ell+1} - (2\ell+1) x P_\ell + \ell P_{\ell-1} = 0$$

see Jackson, Arfken

$\Rightarrow$ can evaluate

$$\int_{-1}^{1} dx\, x P_\ell(x) P_{\ell'}(x) = \text{-----}$$

Associated Legendre Polynomials

$$P_\ell^m = (-1)^m (1-x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} P_\ell(x)$$

$$= 0 \text{ for } m > \ell$$

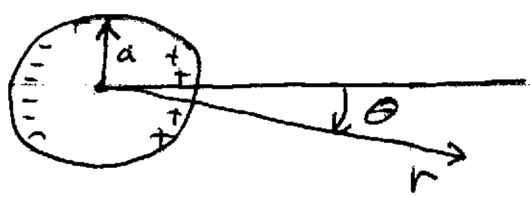
Orthogonality

$$\int_{-1}^{1} dx \, P_\ell^m P_{\ell'}^m = \frac{2}{2\ell+1} \frac{(\ell+m)!}{(\ell-m)!} \delta_{\ell\ell'}$$

$\Rightarrow$ always have the same m

$\Rightarrow$ azimuthal dependences forces $m = m'$

Example Conductor in a uniform external electric field



$\longrightarrow E_0 \quad \Rightarrow \quad \varphi_0 = -E_0 z$

$\qquad\qquad\qquad\qquad = -E_0 r\cos\theta$

$\frac{\partial}{\partial \varphi} = 0 \Rightarrow m=0$

$$\varphi = \sum_\ell \left(a_\ell r^\ell + b_\ell \frac{1}{r^{\ell+1}} \right) P_\ell(\cos\theta)$$

$a_\ell = 0$ for $\ell > 1$ or φ will diverge for large r.

$a_1 = -E_0$ to match external potential

$$\varphi = -E_0 r\cos\theta + \sum_\ell b_\ell \frac{1}{r^{\ell+1}} P_\ell(\cos\theta)$$

Potential independent of θ for $r = a$

$$\varphi(r=a,\theta) = -E_0 a\cos\theta + \sum_\ell b_\ell \frac{1}{a^{\ell+1}} P_\ell(\cos\theta)$$

$$b_\ell = 0 \text{ for } \ell \neq 1$$

$\Rightarrow$ $\cos\theta$ linearly independent of $P_\ell(\cos\theta)$ for $\ell \neq 1$

$\Rightarrow$ ~~could~~ discard $\ell = 0$ since no net charge.

$$\varphi \sim b_0 \frac{1}{r}, \quad E_r \sim -\frac{b_0}{r^2}$$

$$\varphi(r=a,\theta) = \left(-E_0 a + \frac{b_1}{a^2}\right)\cos\theta = 0$$

$$b_1 = E_0 a^3$$

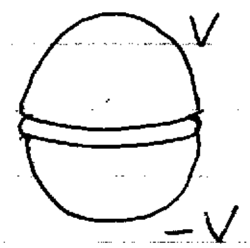
$$\varphi = -E_0 r\cos\theta + \frac{E_0 a^3}{r^2}\cos\theta$$

$$E_r(a) = -\left.\frac{\partial\varphi}{\partial r}\right|_a = E_0\cos\theta + 2\frac{E_0 a^3}{a^3}\cos\theta$$

$$= 3E_0\cos\theta$$

$$\sigma = \epsilon_0 E_r = 3\epsilon_0 E_0\cos\theta$$

Example Hemispherical conductors



$\Rightarrow$ azimuthal symmetry
$m=0$

$$\Phi = \sum_\ell \left(a_\ell r^\ell + b_\ell \frac{1}{r^{\ell+1}}\right) P_\ell(\cos\theta)$$

Odd symmetry around $\theta = \frac{\pi}{2}$ or
$x = \cos\theta = 0 \Rightarrow$ odd function of x
$\Rightarrow \ell$ odd

Since Φ is bounded at ∞, $a_\ell = 0$

$$\Phi = \sum_{\ell\,\text{odd}} b_\ell \frac{1}{r^{\ell+1}} P_\ell(\cos\theta)$$

$$\Phi(a,\theta) = \sum_{\ell\,\text{odd}} b_\ell \frac{1}{a^{\ell+1}} P_\ell(\cos\theta)$$

Multiply by $P_{\ell'}(\cos\theta)$ and integrate to eliminate sum over ℓ

$$\int_{-1}^{1} d\cos\theta\, P_\ell(\cos\theta)\, \Phi(a,\theta) = \frac{2}{2\ell+1} \frac{b_\ell}{a^{\ell+1}}$$

$$V\left[\int_0^1 d\cos\theta\, P_\ell(\cos\theta) - \int_{-1}^{0} d\cos\theta\, P_\ell(\cos\theta)\right]$$

$$= \frac{2}{a^{\ell+1}} \frac{b_\ell}{2\ell+1}$$

Two integrals cancel for ℓ even and add for ℓ odd

$$2V \underbrace{\int_0^1 d\cos\theta \, P_\ell(\cos\theta)} = \frac{2 b_\ell}{a^{\ell+1}(2\ell+1)}$$

$$I \equiv \int_0^1 dx \, P_\ell(x)$$

Use Rodriguez' formula $P_\ell = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell}(x^2-1)^\ell$

$$I = \frac{1}{2^\ell \ell!} \frac{d^{\ell-1}}{dx^{\ell-1}}(x^2-1)^\ell \Big|_0^1$$

$\Rightarrow$ zero for $x = 1$ since

$$\frac{d^{\ell-1}}{dx^{\ell-1}} (x-1)^\ell (x+1)^\ell \sim x-1 \Rightarrow 0$$

$$I = -\frac{1}{2^\ell \ell!} \frac{d^{\ell-1}}{dx^{\ell-1}} (x^2-1)^\ell \Big|_0$$

Use binomial theorem

$$(x^2-1)^\ell = (x^2)^{\ell-q} (-1)^q \frac{\ell!}{q!(\ell-q)!}$$

$\Rightarrow$ summation over q.

Only surviving term has x^0 after $\ell-1$ derivatives

$$2(\ell-q) = \ell-1$$

$$2\ell - 2q = \ell - 1$$

$$q = \frac{\ell+1}{2}$$

$$I = -\frac{1}{2^\ell \ell!}\frac{d^{\ell-1}}{dx^{\ell-1}} \underbrace{x^{2\ell-\ell-1}}_{x^{\ell-1}} (-1)^{\frac{\ell+1}{2}} \frac{\ell!}{\left(\frac{\ell+1}{2}\right)!\left(\frac{\ell-1}{2}\right)!}$$

$$= -\frac{1}{2^\ell}\frac{(-1)^{\frac{\ell+1}{2}}(\ell-1)!}{\left(\frac{\ell+1}{2}\right)!\left(\frac{\ell-1}{2}\right)!} \quad \ell \text{ odd}$$

$$b_\ell = -V \quad a^{\ell+1}\frac{(2\ell+1)(\ell-1)!(-1)^{\frac{\ell+1}{2}}}{2^\ell\left(\frac{\ell+1}{2}\right)!\left(\frac{\ell-1}{2}\right)!}$$

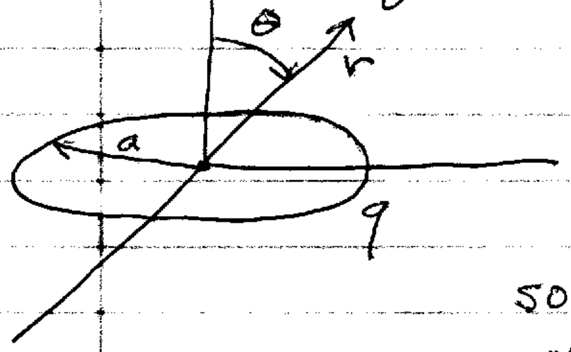
for ℓ odd

$= 0$ for ℓ even

$$\Phi = \sum_{\ell \text{ odd}} b_\ell \frac{P_\ell(\cos\theta)}{r^{\ell+1}}$$

Large r $\quad \Phi \simeq \frac{3}{2} V \frac{a^2}{r^2}\cos\theta$ as before.

<u>Example</u> Ring of charge: solving Poisson's equ. in spherical coordinates



$$\nabla^2 \Phi = -\frac{\rho}{\epsilon_0}$$

Need to express $\rho(r,\theta)$ so that the total integrated charge is q.

$$q = \int d\underset{\sim}{x}\, \rho = 2\pi \int_0^\infty dr \int_{-1}^{1} d(\cos\theta)\, \rho(r,\theta)\, r^2$$

$$\rho = q \frac{1}{2\pi r^2} \delta(\cos\theta)\, \delta(r-a)$$

$$dx = r^2 d\varphi\, d(\cos\theta)\, dr$$

$$\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}\Phi + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}\Phi$$

$$= -\frac{q}{\epsilon_0}\frac{1}{2\pi r^2}\delta(\cos\theta)\,\delta(r-a)$$

Since $\frac{\partial}{\partial\varphi} = 0$,

$$\Phi = \sum_\ell g_\ell(r) P_\ell(\cos\theta)$$

recall
$\Rightarrow$ $$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}P_\ell(\cos\theta) = -\ell(\ell+1)P_\ell(\cos\theta)$$

$$\sum_\ell P_\ell(\cos\theta)\left[\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}g_\ell - \ell(\ell+1)\frac{g_\ell}{r^2}\right]$$

$$= -\frac{q}{\epsilon_0}\;\frac{\delta(\cos\theta)\,\delta(r-a)}{2\pi r^2}$$

Multiply by $P_{\ell'}(\cos\theta)$ and integrate over $\cos\theta$ to eliminate sum over ℓ

$$\sum_\ell \underbrace{\int_{-1}^{1} d(\cos\theta)\,P_{\ell'}(\cos\theta)P_\ell(\cos\theta)}_{\frac{2}{2\ell'+1}\delta_{\ell\ell'}}\left[\quad\right] = -\int_{-1}^{1} d(\cos\theta)\,\delta(\cos\theta)$$

$$\otimes\; P_{\ell'}(\cos\theta)\frac{q}{\epsilon_0}\frac{\delta(r-a)}{2\pi r^2}$$

$\Rightarrow$ yields equation in r

$$\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}g_{\ell'} - \ell'(\ell'+1)\frac{g_{\ell'}}{r^2} = -\frac{q}{\epsilon_0}\frac{(2\ell'+1)}{4\pi r^2}$$

$$\otimes\; P_{\ell'}(0)\,\delta(r-a)$$

$$\equiv S_{\ell'}\,\delta(r-a)$$

$$S_{\ell'} = -\frac{q}{\epsilon_0}\;\frac{2\ell'+1}{4\pi a^2}\,P_{\ell'}(0)$$

Jump conditions at $r=a$,

$$\frac{\partial}{\partial r}g_\ell\Big|_{a-\epsilon}^{a+\epsilon} = S_\ell$$

$$g_\ell\Big|_{a-\epsilon}^{a+\epsilon} = 0$$

For $a \neq r$, $g_\ell \sim r^\ell, r^{-\ell-1}$

<u>$r > a$</u> $g_\ell \to 0$ as $r \to \infty$

$$g_\ell = c_\ell \left(\frac{a}{r}\right)^{\ell+1}$$

<u>$r < a$</u> g_ℓ bounded at $r = 0$

$$g_\ell = c_\ell \left(\frac{r}{a}\right)^{\ell}$$

$\Rightarrow$ g_ℓ continuous at $r = a$.

$\Rightarrow$ use jump in $\frac{\partial}{\partial r} g_\ell$ to determine c_ℓ

$$\left[-(\ell+1)\frac{1}{a} - \ell\frac{1}{a}\right] c_\ell = S_\ell$$

$$c_\ell = -\frac{a S_\ell}{2\ell+1} = \frac{q}{\epsilon_0}\frac{1}{4\pi a} P_\ell(0)$$

note: $P_\ell(0) = 0$ for ℓ odd

$\Rightarrow$ solution even around $x = \cos\theta = 0$

$$\Phi = \sum_{\ell \text{ even}} \frac{q}{4\pi\epsilon_0} P_\ell(0) P_\ell(\cos\theta) \frac{r_<^\ell}{r_>^{\ell+1}}$$

$r_<$ = smaller of r, a

$r_>$ = larger of r, a

$\Rightarrow$ large r, $\ell = 0$ dominates $\Rightarrow$ $\frac{q}{4\pi\epsilon_0}\frac{1}{r}$ point charge

Spherical harmonics

The basis functions $P_\ell^m(\cos\theta)$ and $e^{\pm im\varphi}$ can be combined into a complete set of functions on the unit sphere

$\Rightarrow$ spherical harmonics Y_ℓ^m

$$Y_\ell^m(\theta,\varphi) = \left(\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}\right)^{1/2} P_\ell^m(\cos\theta)\, e^{im\varphi}$$

Orthogonality:

$$\int_0^{2\pi} d\varphi \int_{-1}^{1} d\cos\theta \; Y^*_{\ell' m'}(\theta,\varphi)\, Y_{\ell m}(\theta,\varphi) = \delta_{\ell\ell'}\,\delta_{mm'}$$

$\Rightarrow$ as discussed previously the $\delta_{mm'}$ arises from the φ integral.

δ-function in spherical coordinates

Can represent $\delta(\varphi-\varphi')\,\delta(\cos\theta-\cos\theta')$ using Y_ℓ^m

$$\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} a_\ell^m\, Y_\ell^m(\theta,\varphi) = \delta(\varphi-\varphi')\,\delta(\cos\theta-\cos\theta')$$

$\Rightarrow$ invert the sum to solve for a_ℓ^m

$\Rightarrow$ mult. by $Y_{\ell'}^{m'*}(\theta,\varphi)$ and integrate over 4π

$$\sum_{\ell=0}^{\infty} \sum_{m} a_\ell^m \int_0^{2\pi} d\varphi \int_{-1}^{1} d\cos\theta \; Y_{\ell'}^{m'*}(\theta,\varphi)\, Y_\ell^m(\theta,\varphi)$$

$$= Y_{\ell'}^{m'*}(\theta',\varphi')$$

$$= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_\ell^m \, \delta_{\ell\ell'} \delta_{mm'}$$

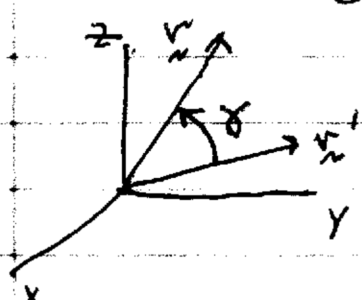
$$= a_{\ell'}^{m'}$$

$$\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_\ell^{m*}(\theta',\varphi')\, Y_\ell^m(\theta,\varphi) = \delta(\varphi-\varphi') \otimes \delta(\cos\theta-\cos\theta')$$

$\Rightarrow$ completeness relation

Infinite medium Greens function in spherical coordinates

$$\nabla^2 G(\mathbf{r},\mathbf{r}') = -4\pi\, \delta(\mathbf{r}-\mathbf{r}')$$



$$\delta(\mathbf{r}-\mathbf{r}') = \frac{1}{r^2}\, \delta(r-r') \otimes \delta(\varphi-\varphi')\, \delta(\cos\theta-\cos\theta')$$

$$\int d\mathbf{r}\, \delta(\mathbf{r}-\mathbf{r}') = \int_0^{\infty} dr \int_0^{2\pi} d\varphi\, r^2 \int_{-1}^{1} d\cos\theta\, \delta = 1$$

$$G(\underset{\sim}{r}, \underset{\sim}{r}') = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} g_\ell^m(r)\, Y_\ell^m(\theta, \varphi)$$

$\Rightarrow$ eliminate sum over ℓ, m using orthogonality

$$\Longrightarrow \int d\varphi\, d\cos\theta\, Y_{\ell'}^{*m'}(\theta, \varphi) [\quad]$$

$$\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} g_\ell(r, r') - \frac{\ell(\ell+1)}{r^2} g_\ell(r, r')$$

$$= -4\pi \frac{1}{r^2} Y_\ell^{m*}(\theta', \varphi')\, \delta(r - r')$$

jump conditions:

$$\left.\frac{\partial g_\ell}{\partial r}\right|_{r'-\epsilon}^{r'+\epsilon} = -4\pi \frac{1}{r'^2} Y_\ell^m(\theta', \varphi')$$

$$\left. g_\ell \right|_{r'-\epsilon}^{r'+\epsilon} = 0$$

For $r \neq r'$:

$$g_\ell \sim r^\ell,\ r^{-\ell-1}$$

$$r > r': \quad g_\ell = c_\ell \left(\frac{r'}{r}\right)^{\ell+1} Y_\ell^{m*}(\theta', \varphi')$$

$$r < r': \quad g_\ell = c_\ell \left(\frac{r}{r'}\right)^{\ell} Y_\ell^{m*}(\theta', \varphi')$$

jump condition for $\frac{\partial g_\ell}{\partial r}$

$$\left(-\frac{(\ell+1)}{r'}-\frac{\ell}{r'}\right)c_\ell = -\frac{4\pi}{r'^2}Y_\ell^{m*}(\theta',\varphi')$$

$$c_\ell = \frac{4\pi}{(2\ell+1)r'}Y_\ell^{m*}(\theta',\varphi')$$

$$G = 4\pi\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{1}{2\ell+1}Y_\ell^{m*}(\theta',\varphi')Y_\ell^m(\theta,\varphi)$$
$$\otimes\ \frac{r_<^{\ell}}{r_>^{\ell+1}}$$

$r_<$ = smaller of r, r'
$r_>$ = larger of r, r'

$\Rightarrow$ infinite medium Greens function

$\Rightarrow$ equivalent to $\frac{1}{|\underset{\sim}{x}-\underset{\sim}{x}'|}$

Alternate forms

For $\theta'=0$, G is independent of φ

$\Rightarrow$ only $m=0$ survives

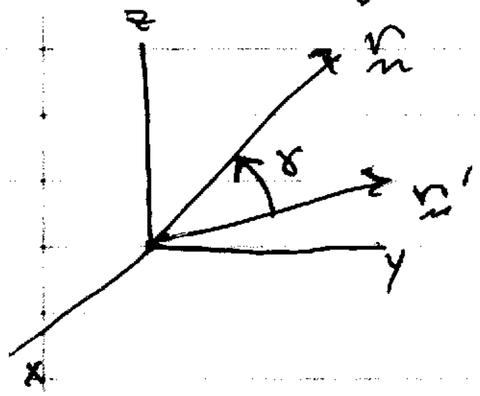
$\Rightarrow P_\ell^m(1)=0$ for $m\neq 0$
(see p. 59)

$P_\ell(1) = 1$ (see p. 58)

$$G = 4\pi \sum_{\ell=0}^{\infty} \frac{1}{2\ell+1} P_\ell(1) P_\ell(\cos\theta) \frac{r_<^\ell}{r_>^{\ell+1}} \frac{2\ell+1}{4\pi}$$

$$G = \sum_{\ell=0}^{\infty} P_\ell(\cos\theta) \frac{r_<^\ell}{r_>^{\ell+1}}$$

$\Rightarrow$ but G only depends on the angle between $\underset{\sim}{r}$, $\underset{\sim}{r}'$



$$G = \sum_{\ell=0}^{\infty} P_\ell(\cos\gamma) \frac{r_<^\ell}{r_>^{\ell+1}}$$

Previously wrote $\cos\gamma$ in terms of θ, φ (p. 38)

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta' \cos(\varphi-\varphi')$$

Comparing G on p. 71 with G here must have

$$P_\ell(\cos\gamma) = 4\pi \sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1} Y_\ell^{m*}(\theta',\varphi') Y_\ell^m(\theta,\varphi)$$

$\Rightarrow$ addition theorem

For $\gamma = 0$, and noting that $P_\ell(1) = 1$

$$1 = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} Y_\ell^{m*}(\theta,\varphi)\, Y_\ell^{m}(\theta,\varphi)$$

$\Rightarrow$ sum rule for spherical harmonics.

<u>Finite medium Greens function</u>

$\Rightarrow$ conducting boundary at $|\mathbf{r}| = a$

$\Rightarrow$ Change the BCs on g_ℓ for $r > r'$ so that $g_\ell = 0$ for $r = a$.