Basis functions for Laplace's and Poisson's eqns: rectangular coordinates

Choose basis functions that satisfy Laplace's eqn

$$\nabla^2 u = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) u = 0$$

$\Rightarrow$ also needed for solving for a Green's function

Assume basis functions are separable

$$u(x,y,z) = X(x)\, Y(y)\, Z(z)$$

$\Rightarrow$ required since orthogonality in any direction can not involve coupling in another direction

$\Rightarrow$ does not imply the solution for u is separable

Substitute u into Laplace's eqn and factor out u

$$u\left(\underbrace{\frac{1}{X}\frac{\partial^2}{\partial x^2}X}_{\text{depends only on } x} + \underbrace{\frac{1}{Y}\frac{\partial^2}{\partial y^2}Y}_{\text{depends only on } y} + \underbrace{\frac{1}{Z}\frac{\partial^2}{\partial z^2}Z}_{\text{depends only on } z}\right) = 0$$

Each term must be a constant for the three to sum to zero.

$$\frac{1}{X}\frac{\partial^2}{\partial x^2}X = -\alpha^2$$

$$\frac{\partial^2}{\partial x^2}X + \alpha^2 X = 0 \Rightarrow X \sim e^{\pm i\alpha x}$$

Similarly, $Y \sim e^{\pm i\beta y}$

For Z, $\frac{1}{Z}\frac{\partial^2}{\partial z^2}Z = \alpha^2+\beta^2$

$\Rightarrow$ couples behavior in z to that in x, y

$\Rightarrow$ required to satisfy Laplace's equ.

$$\Rightarrow Z \sim e^{\pm\sqrt{\alpha^2+\beta^2}\, z}$$

Yields the basis function

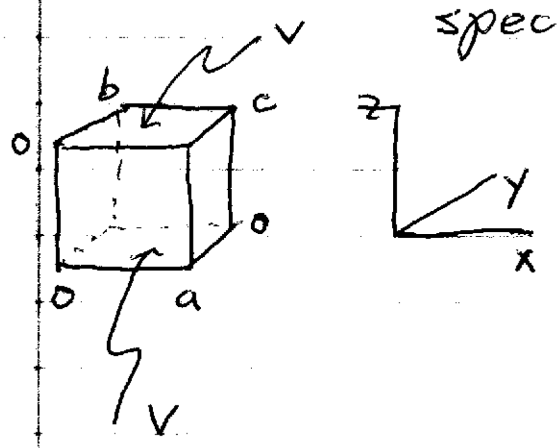
$$u \sim e^{\pm i\alpha x}\, e^{\pm i\beta y}\, e^{\pm(\alpha^2+\beta^2)^{\frac{1}{2}} z}$$

$\Rightarrow$ α, β chosen to satisfy BCs in x, y

$\Rightarrow$ u is oscillatory in two directions but not the third. Why?

$\Rightarrow$ Can permute behavior of x, y, z

<u>Example</u>: Conducting box with potential specified on boundaries



$\Phi = 0$ for $x = 0, a$ and $y = 0, b$

$\Phi = V$ for $z = 0, c$

Have ~~the~~ to match $\Phi = V$ at $z = 0, c$ for all x, y.

$\Rightarrow$ requires oscillatory functions in x, y to do this

$\Rightarrow$ exponential behavior in z

To satisfy BCs at $x = 0, a$ $\Rightarrow \sin\left(\frac{n\pi x}{a}\right)$ n integer

To satisfy BCs at $y = 0, b$ $\Rightarrow \sin\left(\frac{m\pi y}{b}\right)$ m integer

$$\Phi = \sum_{m,n} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \left(A^+_{mn} e^{\gamma_{mn} z} + A^-_{mn} e^{-\gamma_{mn} z}\right)$$

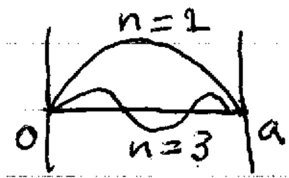
$$\gamma_{mn} = \left(\frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}\right)^{1/2}$$

Use symmetries to simplify solution

$\Rightarrow$ symmetric around $z = \frac{c}{2}$

z dependence $\Rightarrow \cosh[\gamma_{mn}(z-\frac{c}{2})]$

$\Rightarrow$ symmetric around $x = \frac{a}{2} \Rightarrow n$ odd



$\Rightarrow$ symmetric around $y = \frac{b}{2} \Rightarrow m$ odd

$$\varphi = \sum_{\substack{m,n \\ odd}} A_{mn} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \cosh\left[\gamma_{mn}\left(z-\frac{c}{2}\right)\right]$$

BC at $z = c$ (same for $z=0$)

$$V = \sum_{\substack{m,n \\ odd}} A_{mn} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \cosh\left(\gamma_{mn}\frac{c}{2}\right)$$

$\Rightarrow$ invert eqn to solve for A_{mn}

$\Rightarrow$ mult by $\sin\left(\frac{n'\pi x}{a}\right)$ and $\sin\left(\frac{m'\pi y}{b}\right)$

and integrate over x and y

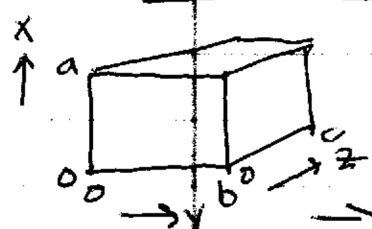
$$\underbrace{V \int_0^a dx \int_0^b dy \, \sin\frac{n'\pi x}{a} \sin\frac{m'\pi y}{b}} = A_{m'n'} \cosh\left(\gamma_{m'n'}\frac{c}{2}\right) \otimes \frac{ab}{4}$$

$$V \frac{a}{n'\pi} \frac{b}{m'\pi} \underbrace{\cos\frac{n'\pi x}{a}\Big|_0^a}_{-2} \underbrace{\cos\frac{m'\pi x}{b}\Big|_0^b}_{-2}$$

$$A_{mn} = \frac{16V}{mn\pi^2} \frac{1}{\cosh\left(\frac{\gamma_{mn} c}{2}\right)}$$

$$\Phi = \frac{16V}{\pi^2} \sum_{\substack{m,n \\ \text{odd}}} \frac{1}{mn} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \times \frac{\cosh \gamma_{mn}\left(z - \frac{c}{2}\right)}{\cosh\left(\gamma_{mn} \frac{c}{2}\right)}$$

$$\gamma_{mn} = \left(\frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}\right)^{1/2}$$



Example : Point charge inside grounded conducting box. Point charge Q at x_0, y_0, z_0

$\Rightarrow$ to solve this problem with image charges would require an infinite number of charges

$$\nabla^2 \Phi = -\frac{Q}{\epsilon_0} \delta(x - x_0)\, \delta(y - y_0)\, \delta(z - z_0)$$

$\Rightarrow$ choose oscillatory functions in x, y and exponential in z,

$\Rightarrow$ arbitrary choice

$$\Phi = \sum_{m,n} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) g_{mn}(z)$$

Substitute into Poisson's eqn

$$\sum_{m,n} \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\left(\frac{\partial^2 g_{mn}}{\partial z^2} - \underset{\text{from } x,y \text{ derivatives}}{\gamma_{mn}^2 g_{mn}}\right)$$

$$= -\frac{Q}{\epsilon_0}\,\delta(x-x_0)\,\delta(y-y_0)\,\delta(z-z_0)$$

$$\gamma_{mn}^2 = \frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}$$

$\Rightarrow$ multiply by $\sin\left(\frac{n'\pi x}{a}\right)$ and $\sin\left(\frac{m'\pi x}{a}\right)$ and integrate over x, y to eliminate sum over m, n.

$$\frac{ab}{4}\left(\frac{\partial^2}{\partial z^2} g_{m'n'} - \gamma_{m'n'}^2\, g_{m'n'}\right)$$

$$= -\frac{Q}{\epsilon_0}\sin\left(\frac{n'\pi x_0}{a}\right)\sin\left(\frac{m'\pi y_0}{b}\right)\delta(z-z_0)$$

$\Rightarrow$ now have an inhomogeneous eqn for $g_{m'n'}$

$\Rightarrow$ solve by solving homogeneous eqn for $z \neq z_0$ and calculating jump conditions across $z = z_0$.

For $z \neq z_0$, (drop primes on m, n)

$$\frac{\partial^2}{\partial z^2} g_{mn} - \gamma_{mn}^2\, g_{mn} = 0$$

$\underline{z < z_0}$

$$g_{mn} = C^{<}_{mn} \sinh(\gamma_{mn} z)$$

$\Rightarrow$ chosen so $\varphi = 0$ at $z = 0$

$\underline{z > z_0}$

$$g_{mn} = C^{>}_{mn} \sinh[\gamma_{mn}(c - z)]$$

$\Rightarrow$ so $\varphi = 0$ at $z = c$

Near $z = z_0$,

$$\frac{\partial^2}{\partial z^2} g_{mn} - \underbrace{\cancel{\gamma_{mn}^2 g_{mn}}}_{\text{small}} = S \;\delta(z - z_0)$$

$$S = -\frac{4Q}{ab\epsilon_0} \sin\frac{n\pi x_0}{a} \sin\frac{m\pi y_0}{b}$$

$\Rightarrow$ when integrating across a δ-function the term with the highest derivative always dominates

$\Rightarrow$ discard $\gamma_{mn}^2 g_{mn}$ near $z = z_0$.

Integrate once,

$$\frac{\partial}{\partial z} g_{mn} \Big|_{z_0 - \epsilon}^{z_0 + \epsilon} = S$$

$\Rightarrow$ jump in slope of g_{mn} at z_0

$\Rightarrow$ g_{mn} is continuous across z_0
since $\frac{\partial}{\partial z} g_{mn}$ is bounded

$$g_{mn} \Big|_{z_0-\epsilon}^{z_0+\epsilon} = 0$$

To have continuity of g_{mn} at z_0, choose

$$C_{mn}^{<} = D_{mn} \sinh\left[\gamma_{mn}(c - z_0)\right]$$

$$C_{mn}^{>} = D_{mn} \sinh(\gamma_{mn} z_0)$$

The jump in slope the becomes

$$-\gamma_{mn} D_{mn} \Big[\cosh\left[\gamma_{mn}(c - z_0)\right] \sinh(\gamma_{mn} z_0)$$

$$+ \cosh(\gamma_{mn} z_0) \sinh\left[\gamma_{mn}(c - z_0)\right] \Big]$$

$$= S'$$

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\Rightarrow \; -\gamma_{mn} D_{mn} \sinh\left[\gamma_{mn}(c - z_0 + z_0)\right] = S'$$

$$D_{mn} = \frac{S'}{-\gamma_{mn} \sinh(\gamma_{mn} c)}$$

$$\Phi = \frac{4Q}{ab\epsilon_0} \sum_{m,n} \frac{1}{\gamma_{mn}\sinh(\gamma_{mn}c)}$$

$$\otimes \sin\left(\frac{n\pi x_0}{a}\right)\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y_0}{b}\right)\sin\left(\frac{m\pi y}{b}\right)$$

$$\otimes \sinh(\gamma_{mn} z_<)\sinh\left[\gamma_{mn}(c - z_>)\right]$$

$z_<$ = smaller of z, z_0

$z_>$ = larger of z, z_0