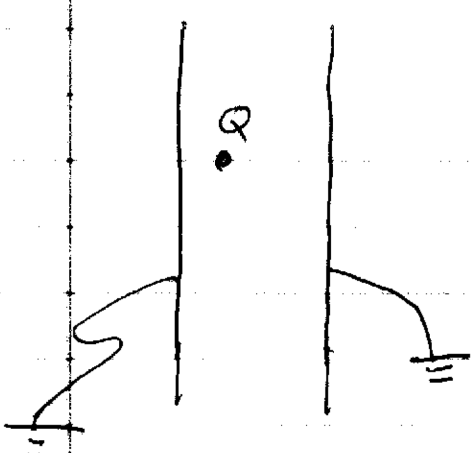
Solving Poisson's and Laplace's eqn using orthogonal basis functions

When the geometry of a problem with finite BC's is simple, can ~~be~~ use the method of images to construct ~~solutions of~~ Green's functions. In complex geometries this is often difficult.



Even two plane conductors with a charge between requires an infinite # of image charges.

Another approach is to expand the solutions of Poisson's or Laplace's eqns using orthogonal functions. Need to tailor the basis functions to fit the geometry of the problem

$\Rightarrow$ can't use a traditional Fourier representation in a cylindrical or spherical system.

Orthogonal functions from Stürm-Liouville operators (e.g. Arfken Ch. 8)

A S-L equation takes the following form:

$$\frac{\partial}{\partial x} P(x) \frac{\partial}{\partial x} u_n + q(x)\, u_n + w(x)\, \lambda_n u_n = 0$$

where $p(x)$ is positive except possibly at the endpoints of the region of interest $x \in (a, b)$. $w(x)$ is the weight function and is positive except at a finite # of points where it can be zero. λ_n is the eigen value of the eigenfunction u_n which satisfies the BC's

$$P(x)\, u_n u_m' \Big|_a^b = 0 .$$

Defining $\mathcal{L} \equiv \frac{\partial}{\partial x} P \frac{\partial}{\partial x} + q$, then

S-L eigenfunctions satisfy the integral relation

$$\int_a^b dx\, u_n^* (\mathcal{L}\, u_m) = \int_a^b dx\, u_m (\mathcal{L}\, u_n)^* .$$

Eigenfunctions of S-L operators have the following properties:

1. Eigenvalues are real
2. Non-degenerate eigenfunctions are orthogonal
3. The eigenfunctions form a complete set.

Representing functions with orthogonal functions

Consider S-L eigenfunctions defined over the interval (a, b) such that

$$\int_a^b dx\, w(x)\, u_n^*(x)\, u_m(x) = \delta_{mn}$$

The u_n are an orthonormal set. Consider a function $f(x)$ over (a, b). We can write

$$f(x) = \sum_{n=0}^{\infty} a_n u_n(x)$$

With a_n calculated from the orthogonality relation

$$a_n = \int_a^b dx\, w(x)\, f(x)\, u_n^*(x)$$

The expansion is useful if $a_n \to 0$ as $n \to \infty$ and in particular if

$$f_N \equiv \sum_{n=0}^{N} a_n u_n(x)$$

can be made arbitrarily small as $N \to \infty$.

Example: Solutions of 1-D Helmholtz equ over $x \in (a, b)$ with BC's zero at a, b.

$$\frac{d^2}{dx^2} u_n + k_n^2 u_n = 0$$

$\Rightarrow$ S-L form with proper BCs

$\Rightarrow p(x) = 1, \; q(x) = 0, \; w = 1$

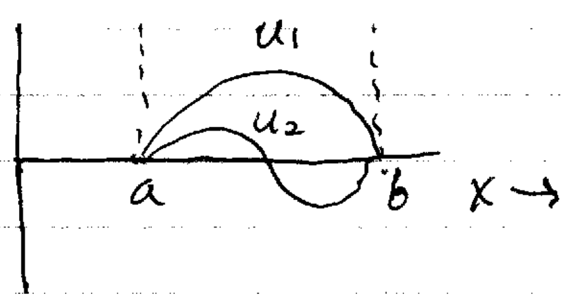
Solutions: $\sin(k_n x)$, $\cos(k_n x)$

$\Rightarrow \sin k_n(x-a) \Rightarrow$ zero at $x = a$

$\Rightarrow u_n(b) = 0 \Rightarrow k_n(b-a) = n\pi$

$$k_n = \frac{n\pi}{b-a}$$

$\Rightarrow u_n \sim$ ~~[illegible]~~ $\sin\left[\frac{n\pi(x-a)}{b-a}\right]$



Basis functions are oscillatory

$$f(x) = \sum_n a_n u_n(x)$$

What if $b-a \to \infty$?

$$K_n = \frac{n\pi}{b-a} \qquad dK = \frac{\pi}{b-a}\,dn$$

$$\sum_n \Rightarrow \int dn = \frac{b-a}{\pi}\int dK$$

2-D system: $f(x,y) = \sum_{m,n} a_{mn} U_m(x) V_n(y)$

<u>Choice of basis functions for Laplace's eqn</u>

$$\nabla^2 \varphi = 0, \quad \varphi = \sum_{l,m,n} C_{lmn} U_{lmn}(x,y,z)$$

$\Rightarrow$ choose basis functions that satisfy Laplaces eqn,

$$\nabla^2 U_{lmn} = 0$$

$\Rightarrow$ match BC's

$\Rightarrow$ exploit symmetry to eliminate basis functions