Green's Function Solutions of Poisson's Eqn.

We have shown previously that the solution of Poisson's eqn.

$$\nabla^2 \Phi = -\frac{\rho}{\epsilon_0}$$

in free space is given by

$$\Phi(\underset{\sim}{x}) = \frac{1}{4\pi\epsilon_0}\int d\underset{\sim}{x}' \frac{\rho(\underset{\sim}{x}')}{|\underset{\sim}{x}-\underset{\sim}{x}'|}$$

This is because $\frac{1}{|\underset{\sim}{x}-\underset{\sim}{x}'|}$ is the Green's function of ~~Laplace~~ Poisson's eqn with a δ-function source in free space,

$$\nabla^2 \frac{1}{|\underset{\sim}{x}-\underset{\sim}{x}'|} = -4\pi\,\delta(\underset{\sim}{x}-\underset{\sim}{x}').$$

More generally, we can have a system with conductors where Φ or $\underset{\sim}{E}\cdot\hat{n}$ are specified rather than simply $\Phi \to 0$ as $\underset{\sim}{x} \to \infty$. Can we construct a Green's function to enable us to solve such problems?

$$\varphi(\underset{\sim}{x}) = \int d\underset{\sim}{x}' \, \delta(\underset{\sim}{x}-\underset{\sim}{x}') \, \varphi(\underset{\sim}{x}')$$

Let $\nabla'^2 G(\underset{\sim}{x},\underset{\sim}{x}') = -4\pi \, \delta(\underset{\sim}{x}-\underset{\sim}{x}')$

G is the response at $\underset{\sim}{x}'$ from a source at $\underset{\sim}{x}$.

$$\varphi(\underset{\sim}{x}) = -\frac{1}{4\pi}\int d\underset{\sim}{x}' \left[\nabla'^2 G(\underset{\sim}{x},\underset{\sim}{x}')\right] \varphi(\underset{\sim}{x}')$$

But

$$\nabla' \cdot \left(\varphi(\underset{\sim}{x}')\nabla' G\right) = \nabla'\varphi \cdot \nabla' G + \varphi(\underset{\sim}{x}')\nabla'^2 G$$

$$\Rightarrow \quad = -\frac{1}{4\pi}\underbrace{\int d\underset{\sim}{x}' \, \nabla' \cdot \left(\varphi(\underset{\sim}{x}')\nabla' G\right)}_{\text{use divergence theorem}} + \frac{1}{4\pi}\underbrace{\int d\underset{\sim}{x}' \, \nabla'\varphi \cdot \nabla' G}_{\text{again write as divergence}}$$

$$= -\frac{1}{4\pi}\int ds' \, \varphi(\underset{\sim}{x}') \frac{\partial}{\partial n'} G$$

$$+ \frac{1}{4\pi}\int d\underset{\sim}{x}' \left[\underbrace{\nabla' \cdot G \nabla'\varphi}_{\substack{\text{divergence} \\ \text{theorem}}} - \underbrace{G \nabla'^2 \varphi}_{-\rho(\underset{\sim}{x}')\frac{1}{\epsilon_0}}\right]$$

where $\frac{\partial}{\partial n'} = \hat{n}' \cdot \nabla$

$$\Phi(\underset{\sim}{x}) = \frac{1}{4\pi\epsilon_0} \int_V d\underset{\sim}{x}' \, G(\underset{\sim}{x}, \underset{\sim}{x}') \, \rho(\underset{\sim}{x}')$$

$$+ \frac{1}{4\pi} \int_S ds' \left[G(\underset{\sim}{x}, \underset{\sim}{x}') \frac{\partial \Phi}{\partial n'} - \Phi(\underset{\sim}{x}') \frac{\partial G}{\partial n'} \right]$$

This is completely general. In infinite system where $S \to \infty$ and $G = 1/|\underset{\sim}{x} - \underset{\sim}{x}'|$ same as earlier result.

Dirichlet BC's specify Φ on S:
Choose $G(\underset{\sim}{x}, \underset{\sim}{x}') = 0$ on S so that

$$\Phi(\underset{\sim}{x}) = \frac{1}{4\pi\epsilon_0} \int_V d\underset{\sim}{x}' \, G(\underset{\sim}{x}, \underset{\sim}{x}') \, \rho(\underset{\sim}{x}')$$

$$- \frac{1}{4\pi} \int ds' \, \Phi(\underset{\sim}{x}') \frac{\partial}{\partial n'} G$$

Neumann BC's specify $\partial\Phi/\partial n'$ on S:
Would like to choose $\partial G/\partial n' = 0$.
However, this is not possible since

$$\int_S ds' \frac{\partial G}{\partial n'} = \int_V d\underset{\sim}{x}' \, \nabla' \cdot (\nabla' G)$$

$$= -4\pi \int_V d\underset{\sim}{x}' \, \delta(\underset{\sim}{x} - \underset{\sim}{x}') = -4\pi$$

Best option is to choose $\frac{\partial}{\partial n'} G = -\frac{4\pi}{S}$

where $S = \int_S ds'$

Then,

$$\Phi(\underset{\sim}{x}) = \frac{1}{4\pi\epsilon_0} \int d\underset{\sim}{x}' \, G(\underset{\sim}{x}, \underset{\sim}{x}') \, \rho(\underset{\sim}{x}')$$

$$+ \frac{1}{4\pi} \int ds' \, G(\underset{\sim}{x}, \underset{\sim}{x}') \frac{\partial \Phi}{\partial n'}$$

$$+ \underbrace{\frac{1}{S} \int ds' \, \Phi(\underset{\sim}{x}')}_{\langle \Phi \rangle_S \,=\, \text{average over } S \text{ of } \Phi}$$

$\langle \Phi \rangle_S$ typically zero if some part of surface is at ∞.

Conclusion: Have reduced the problem of solving $\nabla^2 \Phi = -\rho/\epsilon_0$ to solving

$$\nabla^2 G = -4\pi \, \delta(\underset{\sim}{x} - \underset{\sim}{x}')$$

with simple BC's.

<u>Greens function for a sphere with Dirichlet BCs</u>

$G(\underset{\sim}{x},\underset{\sim}{x}')$ satisfies

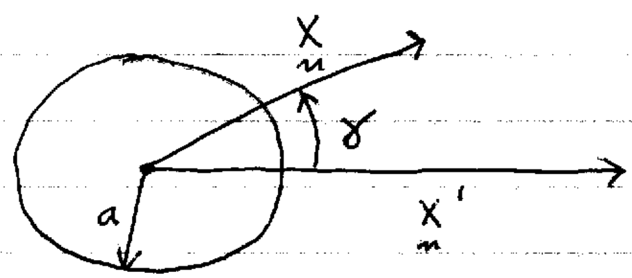
$$\nabla^2 G(\underset{\sim}{x},\underset{\sim}{x}') = -4\pi\,\delta(\underset{\sim}{x}-\underset{\sim}{x}')$$

For Dirichlet BC want $G(\underset{\sim}{x},\underset{\sim}{x}') = 0$ for $\underset{\sim}{x}$ on the surface. We previously calculated Φ for a point charge outside a grounded sphere. For that problem

$$\nabla^2 \Phi(\underset{\sim}{x}) = -\frac{q}{\epsilon_0}\,\delta(\underset{\sim}{x}-\underset{\sim}{y})$$

Let $\underset{\sim}{y} \rightarrow \underset{\sim}{x}'$ and $\frac{q}{4\pi\epsilon_0} = 1$. This yields

$$G(\underset{\sim}{x},\underset{\sim}{x}') = \frac{1}{|\underset{\sim}{x}-\underset{\sim}{x}'|} - \frac{a}{|\underset{\sim}{x}'|}\,\frac{1}{\left|\underset{\sim}{x}-\frac{a^2}{|\underset{\sim}{x}'|^2}\underset{\sim}{x}'\right|}$$



$$|\underset{\sim}{x}-\underset{\sim}{x}'| = \left[(\underset{\sim}{x}-\underset{\sim}{x}')\cdot(\underset{\sim}{x}-\underset{\sim}{x}')\right]^{\frac{1}{2}} = \left[x^2+x'^2-2xx'\cos\gamma\right]^{\frac{1}{2}}$$

$$G = \frac{1}{\left[x^2 + x'^2 - 2xx'\cos\gamma\right]^{1/2}} - \frac{a}{\left[x^2x'^2 + a^4 - 2xx'a^2\cos\gamma\right]^{1/2}}$$

$\Rightarrow$ note that $G(\underset{\sim}{x}, \underset{\sim}{x}')$ is symmetric when $\underset{\sim}{x} \longleftrightarrow \underset{\sim}{x}'$

To calculate $\Phi(\underset{\sim}{x})$ also need $\frac{\partial G}{\partial n'} = -\left.\frac{\partial G}{\partial x'}\right|_{x'=a}$

$$\frac{\partial G}{\partial x'} = -\frac{1}{2}\, 2\, \frac{(x' - x\cos\gamma)}{|\underset{\sim}{x} - \underset{\sim}{x}'|^3} + \frac{1}{2}\, a\, \frac{2\,(x'x^2 - xa^2\cos\gamma)}{(x^2x'^2 + a^4 - 2xx'a^2\cos\gamma)^{3/2}}$$

$$\left.\frac{\partial G}{\partial x'}\right|_{x'=a} = -\frac{a - x\cos\gamma}{|\underset{\sim}{x} - a\hat{n}|^3} + \frac{a^2}{a^3}\, \frac{(x^2 - xa\cos\gamma)}{\underbrace{(x^2 + a^2 - 2xa\cos\gamma)^{3/2}}_{|\underset{\sim}{x} - \hat{n}a|^3}}$$

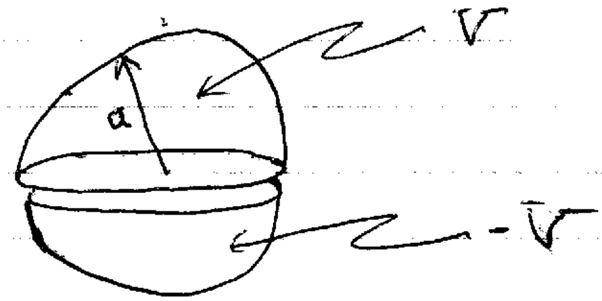
$$= \frac{1}{a\,|\underset{\sim}{x} - a\hat{n}|^3}\left[-a^2 + \cancel{ax\cos\gamma} + x^2 - \cancel{xa\cos\gamma}\right]$$

Note that $\hat{n}'$ is actually inward since the volume V is outside the sphere

$$\frac{\partial G}{\partial n'} = -\left.\frac{\partial G}{\partial x'}\right|_{x'=a} = -\frac{(x^2 - a^2)}{a\,|\underset{\sim}{x} - a\hat{n}|^3}$$

$$\Phi(\underset{\sim}{x}) = \int_V d\underset{\sim}{x}'\, \rho(\underset{\sim}{x}')\, G(\underset{\sim}{x}, \underset{\sim}{x}') - \frac{1}{4\pi}\oint_S ds'\, \Phi(\underset{\sim}{x}')\, \frac{\partial G}{\partial n'}$$

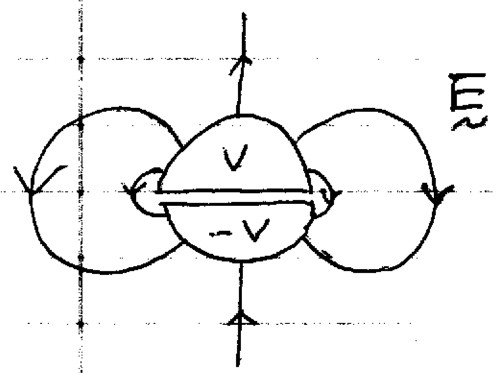
Hemisphere's at different potentials



No change outside

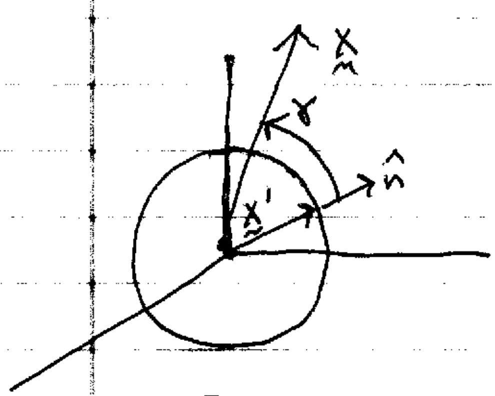
$\rho(\underset{\sim}{x}) = 0$ for

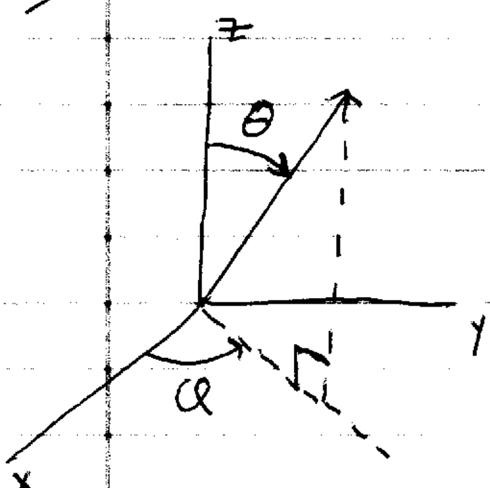
$|\underset{\sim}{x}| > a$



$$\Phi(\underset{\sim}{x}) = -\int \frac{ds'}{4\pi}\, \Phi(\underset{\sim}{x}')\, \frac{\partial G}{\partial n'}$$

$$\left.\frac{\partial G}{\partial n'}\right|_a = -\frac{(x^2 - a^2)}{a\,|\underset{\sim}{x} - a\hat{n}|^3}$$





$$\frac{\underset{\sim}{x}}{|\underset{\sim}{x}|} = \hat{i}\,\sin\theta\cos\varphi + \hat{j}\,\sin\theta\,\sin\varphi + \hat{k}\,\cos\theta$$

$\Rightarrow$ similar for $\hat{n}\,(\theta', \varphi')$

φ here is azimuthal angle

$$\frac{\underset{\sim}{x}\cdot\hat{n}}{|\underset{\sim}{x}|} = \cos\gamma = \sin\theta\cos\varphi\sin\theta'\cos\varphi' + \sin\theta\sin\varphi\sin\theta'\sin\varphi' + \cos\theta\cos\theta'$$

identity: $\cos\varphi\cos\varphi' + \sin\varphi\sin\varphi' = \cos(\varphi-\varphi')$

$$\cos\gamma = \sin\theta\sin\theta'\cos(\varphi-\varphi') + \cos\theta\cos\theta'$$

$\Rightarrow$ integrate over surface of sphere with

$$ds' = a^2\, d\cos\theta'\, d\varphi'$$

$\cos\theta' \in (-1,1)$ and $\varphi' \in (0,2\pi)$

$\Rightarrow$ Can't do surface integral exactly

$\Rightarrow$ Consider case with $\theta = 0$

$\Rightarrow$ along the axis of the sphere

$\Rightarrow \cos\gamma = \cos\theta'$

$\Rightarrow |\underset{\sim}{x}| = z$

$$\frac{\partial G}{\partial n'} = -\frac{(z^2-a^2)}{a(z^2+a^2-2za\cos\theta')^{3/2}}$$

$$\Phi = -\frac{1}{4\pi} a^2 V \, 2\pi \left[\int_0^1 d\cos\theta' - \int_{-1}^0 d\cos\theta' \right] \frac{\partial G}{\partial n'}$$

$$= + \frac{\cancel{a^2} V}{2} \frac{(z^2-a^2)}{-2z\cancel{a^2}} \left[\int_0^1 d\cos\theta' - \int_{-1}^0 d\cos\theta' \right] \frac{-1za}{(z^2+a^2-2za\cos\theta')^{3/2}}$$

$$= \frac{V(z^2-a^2)}{-4z} \left[\frac{-2}{(z^2+a^2-2za\cos\theta')^{\frac{1}{2}}} \right] \left[\Big|_0^1 - \Big|_{-1}^0 \right]$$

$$= \frac{V}{2z}(z^2-a^2) \left[\frac{1}{(z^2+a^2-2za)^{1/2}} - \frac{2}{(z^2+a^2)^{1/2}} + \frac{1}{(z^2+a^2+2za)^{1/2}} \right]$$

$$= \frac{V}{2z}(z^2-a^2) \left[\frac{1}{z-a} - \frac{2}{(z^2+a^2)^{1/2}} + \frac{1}{z+a} \right]$$

$$= \frac{V}{\cancel{2z}}(z^2-a^2) \left[\frac{2z}{z^2-a^2} - \frac{2}{(z^2+a^2)^{\frac{1}{2}}} \right]$$

$$\Phi = V \left[1 - \frac{z^2-a^2}{z(z^2+a^2)^{1/2}} \right] \quad \text{for } z > a.$$

As $z \to a \Rightarrow \Phi \to V$

~~As z~~ For $z \gg a$

$$\Phi \simeq V \left[1 - \frac{\cancel{z^2}\left(1-\frac{a^2}{z^2}\right)\left(1-\frac{1}{2}\frac{a^2}{z^2}\right)}{\cancel{z^2}} \right]$$

$$= V \left[1 - 1 + \frac{a^2}{z^2} + \frac{1}{2}\frac{a^2}{z^2} \right] = \frac{3}{2} V \frac{a^2}{z^2}$$

$\Rightarrow$ Falls off faster than $\frac{1}{z}$. Why?

$\Rightarrow$ Consider large $|\underset{\sim}{x}| \equiv r \gg a$
with arbitrary angle θ

$$\frac{\partial G}{\partial n'} = -\frac{(r^2 - a^2)}{a(r^2 + a^2 - 2ar\cos\gamma)^{3/2}}$$

$$= -\frac{r^2\left(1 - \frac{a^2}{r^2}\right)}{a r^3 \left(1 + \frac{a^2}{r^2} - \frac{2a}{r}\cos\gamma\right)^{3/2}}$$

$$\simeq -\frac{1}{ar}\left(1 + 3\frac{a}{r}\cos\gamma\right)$$

$$\Phi = -\frac{a^2 V}{4\pi}\left(-\frac{1}{ar}\right)\int_0^{2\pi} d\phi' \left[\int_0^1 d\cos\theta' - \int_{-1}^0 d\cos\theta'\right]$$

$$\otimes \Big[\underbrace{1}_{\substack{\text{zero when} \\ \text{averaged over} \\ \theta'}} + 3\frac{a}{r}\big(\underbrace{\sin\theta\sin\theta'\cos(\phi-\phi')}_{\substack{\text{periodic in } \phi' \\ \Rightarrow \text{zero average}}} + \cos\theta\cos\theta'\big)\Big]$$

$$= \frac{3a^2 V}{2r^2}\cos\theta \underbrace{\left[\int_0^1 d\cos\theta' - \int_{-1}^0 d\cos\theta'\right]}_{2\int_0^1 d\cos\theta'} \cos\theta'$$

$$\Phi = \frac{3}{2}\frac{a^2}{r^2} V \cos\theta \quad \Rightarrow \text{dipole field}$$