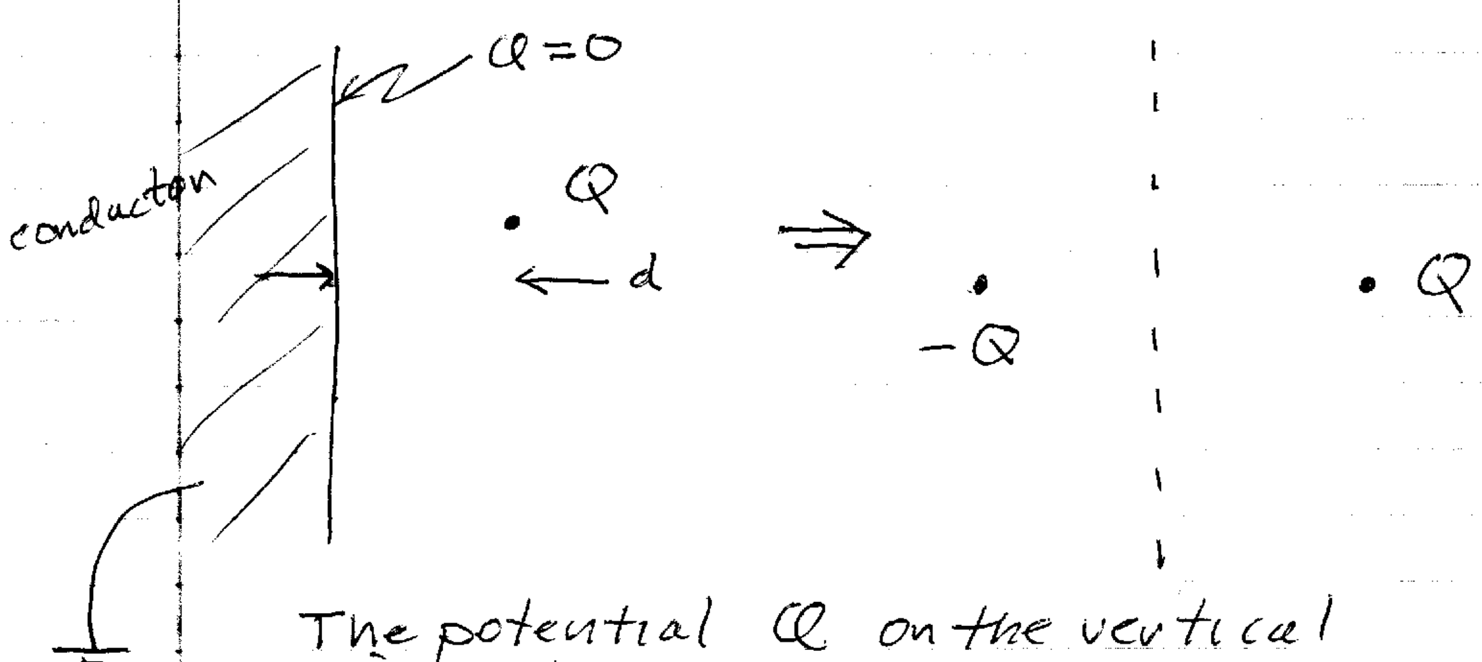
Method of images

Suppose you have a charge or group of charges in the vicinity of conductors with some specified boundary conditions. The BCs can be matched by placing image charges.

$\Rightarrow$ uniqueness theorem guarantees that you have the correct solution

<u>Example</u> Infinite plane conductor with charge Q



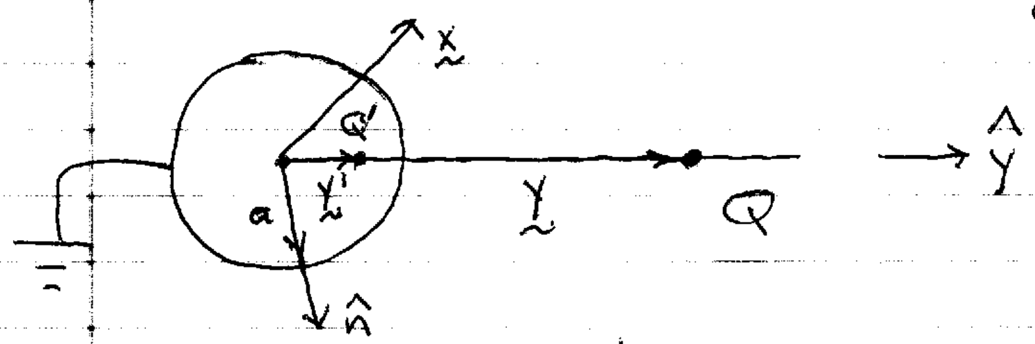
The potential ϕ on the vertical line between the two charges yields the same BC's as the conducting plane. The potential on the right of the dashed line is the same as with plate. Why?

Can calculate force of Q, surface charge density on the conductor,

$$F = \frac{Q^2}{4\pi\epsilon_0 (2d)^2} \quad \text{(to the left)}$$

Charge density from $E_n = \frac{\sigma}{\epsilon_0}$ with E_n along dashed line.
Integrated value of charge on conductor?

<u>Example</u> Grounded conducting sphere with external charge



Zero potential on sphere $\Rightarrow \varphi(|\underset{\sim}{x}| = a) = 0$

Choose an image charge Q' inside the spherical surface along $\underset{\sim}{y}$ axis at location $\underset{\sim}{y}'$.

$$\varphi(\underset{\sim}{x}) = \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{|\underset{\sim}{x} - \underset{\sim}{y}|} + \frac{Q'}{|\underset{\sim}{x} - \underset{\sim}{y}'|}\right]$$

At $|\underset{\sim}{x}| = a$, $\underset{\sim}{x} = a\hat{n}$

$$0 = \Phi(|\underset{\sim}{x}| = a) = \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{|a\hat{n} - y\hat{y}|} + \frac{Q'}{|a\hat{n} - y'\hat{y}|}\right]$$

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{a}\,\frac{1}{|\hat{n} - \frac{y}{a}\hat{y}|} + \frac{Q'}{y'}\,\frac{1}{|\hat{y} - \frac{a}{y'}\hat{n}|}\right]$$

By inspection,

$$\frac{Q}{a} = -\frac{Q'}{y'}\ , \qquad \underbrace{\frac{y}{a} = \frac{a}{y'}}$$

$$\boxed{Q' = -Q\,\frac{a}{y}} \qquad \boxed{y' = \frac{a^2}{y}}$$

$Q' < Q$ $\qquad\qquad$ $\Rightarrow y' < a$

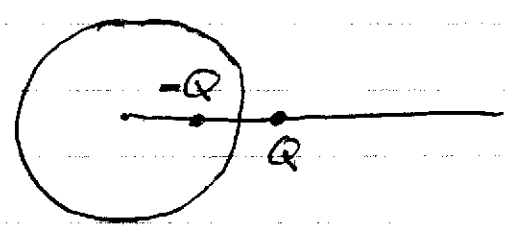
Note: As y decreases, Q' increases and y' increases.

In the limit $y = a + \epsilon$ with $\epsilon \to 0$,

$$y' = \frac{a^2}{a+\epsilon} = a\,\frac{1}{1+\frac{\epsilon}{a}} \simeq a\left(1 - \frac{\epsilon}{a}\right) = a - \epsilon$$

and $Q' = -Q\,\dfrac{a}{a+\epsilon} \simeq -Q$

What is happening?



Surface charge σ on sphere:

$$E_n = \frac{\sigma}{\epsilon_0} \implies \text{calculate } E_n \text{ at } |\mathbf{x}| = a$$

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{|\mathbf{x}-\mathbf{y}|} - \frac{Qa}{y}\frac{1}{\left|\mathbf{x} - \frac{a^2}{y^2}\mathbf{y}\right|}\right]$$

$$\mathbf{E} = -\nabla\Phi = \frac{1}{4\pi\epsilon_0}\left[\frac{Q(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3} - \frac{Qa}{y}\frac{\left(\mathbf{x} - \frac{a^2}{y^2}\mathbf{y}\right)}{\left|\mathbf{x} - \frac{a^2}{y^2}\mathbf{y}\right|^3}\right]$$

$$\implies \mathbf{x} = a\hat{n}$$

$$E_n = \hat{n}\cdot\mathbf{E}$$

$$E_n = \frac{Q}{4\pi\epsilon_0}\left[\frac{a - y\,\hat{n}\cdot\hat{y}}{a^3\left|\hat{n} - \frac{y}{a}\hat{y}\right|^3} - \frac{a}{y}\frac{\left(a - \frac{a^2}{y}\hat{n}\cdot\hat{y}\right)}{a^3\left|\hat{n} - \frac{a}{y}\hat{y}\right|^3}\right]$$

$$= \frac{Q}{4\pi\epsilon_0 a^3}\left[\frac{a - y\,\hat{n}\cdot\hat{y}}{\left|\hat{n} - \frac{y}{a}\hat{y}\right|^3} - \frac{a}{y}\frac{y^3}{a^3}\,a\,\frac{\left(1 - \frac{a}{y}\hat{n}\cdot\hat{y}\right)}{\left|\hat{y} - \frac{y}{a}\hat{n}\right|^3}\right]$$

$$= \frac{Q}{4\pi\epsilon_0 a^3}\frac{1}{\left|\hat{n} - \frac{y}{a}\hat{y}\right|^3}\left[a - y\,\hat{n}\cdot\hat{y} - \frac{y^2}{a} + y\,\hat{n}\cdot\hat{y}\right]$$

$$E_n = \frac{Q}{4\pi\epsilon_0 a^2}\frac{1}{\left|\hat{n} - \frac{y}{a}\hat{y}\right|^3}\left(1 - \frac{y^2}{a^2}\right)$$

$\sigma = \epsilon_0 E_n \Rightarrow E_n < 0 \Rightarrow \sigma < 0$
for $Q > 0$.

$\Rightarrow \sigma$ largest when $\hat{n} = \hat{y}$
σ smallest when $\hat{n} = -\hat{y}$

$\Rightarrow$ integral over σ must be Q'.

Force between charge and sphere:

Easiest to calculate $\underset{\sim}{E}$ at Q due to Q'

$$\underset{\sim}{E}_{Q'}(\underset{\sim}{x} = y\hat{y}) = -\frac{1}{4\pi\epsilon_0}\frac{Qa}{y}\frac{\left(y - \frac{a^2}{y}\right)\hat{y}}{\left(y - \frac{a^2}{y}\right)^3}$$

$$E_y = -\frac{1}{4\pi\epsilon_0}\frac{Qa}{y}y^2\frac{1}{(y^2-a^2)^2}$$

$$F_y = QE_y$$

$$F_y = -\frac{1}{4\pi\epsilon_0}\frac{Q^2ay}{(y^2-a^2)^2}$$

As $y \to \infty$,

$$F_y \sim -\frac{1}{4\pi\epsilon_0}\frac{Q^2a}{y^3}$$

Why?