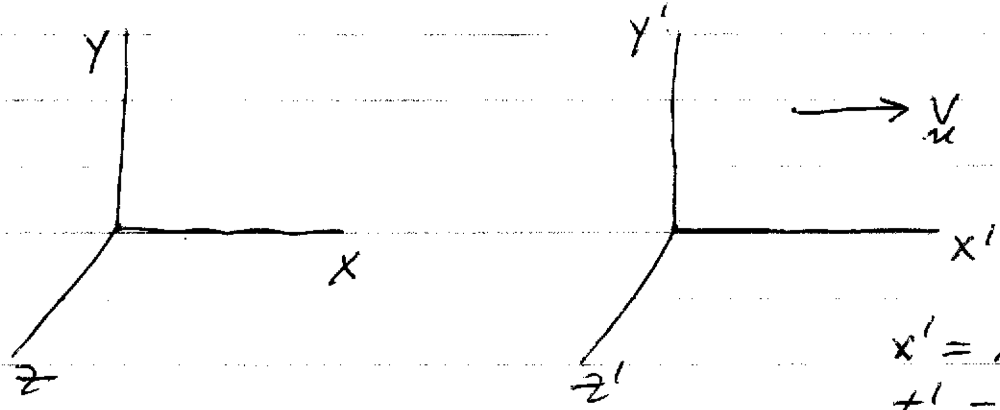
Special Relativity

Early after the formulation of Maxwell's eqns it was shown that the form of the equations was not invariant under a Galilean transformation



$$x' = x - vt$$
$$t' = t$$
$$y' = y$$
$$z' = z$$

The wave equation in the $\underset{\sim}{x}'$ frame is given by

$$\left(\nabla'^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}\right) \underset{\sim}{B} = 0$$

$$\nabla'^2 = \frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2}$$

$$\frac{\partial}{\partial x'} = \frac{\partial x}{\partial x'}\frac{\partial}{\partial x} + \cancelto{0}{\frac{\partial t}{\partial x'}}\frac{\partial}{\partial t}$$

$$\frac{\partial}{\partial t'} = \frac{\partial x}{\partial t'}\frac{\partial}{\partial x} + \frac{\partial t}{\partial t'}\frac{\partial}{\partial t}$$

$$x(x', t') = x' + vt'$$

$$\frac{\partial x}{\partial t'} = v \,, \quad \frac{\partial t}{\partial t'} = 1$$

$$\frac{\partial}{\partial t'} = v\frac{\partial}{\partial x} + \frac{\partial}{\partial t}$$

Thus, the wave equation in the $\underset{\sim}{x}$ frame should be

$$\left[\nabla^2 - \frac{1}{c^2}\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)^2\right]\underset{\sim}{B} = 0$$

This is clear not consistent with Maxwell's eqns.

It was shown by Lorentz that Maxwell's equations are invariant under another set of transformations

$$\left.\begin{aligned} ct' &= \gamma\,(ct - \beta x) \\ x' &= \gamma(x - \beta ct) \\ y' &= y \\ z' &= z \end{aligned}\right\}\ \text{Lorentz transformation}$$

with $\gamma = \frac{1}{(1-\beta^2)^{1/2}}$, $\beta = \frac{v}{c}$

$$\frac{\partial}{\partial x'} = \frac{\partial x}{\partial x'}\frac{\partial}{\partial x} + \frac{\partial t}{\partial x'}\frac{\partial}{\partial t}$$

$$x = \gamma(x' + \beta ct')$$

$$ct = \gamma(ct' + \beta x')$$

$$\frac{\partial}{\partial x'} = \gamma\frac{\partial}{\partial x} + \frac{\gamma\beta}{c}\frac{\partial}{\partial t}$$

$$\frac{1}{c}\frac{\partial}{\partial t'} = \gamma\beta\frac{\partial}{\partial x} + \frac{\gamma}{c}\frac{\partial}{\partial t}$$

Thus, in the moving frame

$$\left(\nabla'^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}\right)\underset{\sim}{B} = 0$$

and can be written ~~in the~~ as

$$\left[\gamma^2 \frac{\partial^2}{\partial x^2} + \cancel{2\frac{\gamma^2\beta}{c}\frac{\partial}{\partial x}\frac{\partial}{\partial t}} + \frac{\gamma^2\beta^2}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \left(\gamma^2\beta^2\frac{\partial^2}{\partial x^2} + \cancel{2\frac{\gamma^2\beta}{c}\frac{\partial}{\partial x}\frac{\partial}{\partial t}} + \frac{\gamma^2}{c^2}\frac{\partial^2}{\partial t^2}\right)\right]\underset{\sim}{B} = 0$$

$$\left[\gamma^2(1-\beta^2)\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{\gamma^2}{c^2}(1-\beta^2)\frac{\partial^2}{\partial t^2}\right]\underset{\sim}{B} = 0$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\underset{\sim}{B} = 0$$

In Maxwell's eqns the wave equation takes the same form in any reference frame. This implies that light propagates with velocity c, independent of the reference frame. This result contradicted the understanding of waves at the time as propagating in a medium (e.g., sound waves). where the propagation velocity is tied to the medium.

⟹ Physicists assumed that Maxwell's eqns were wrong

Experiments carried out by Michelson-Morly to detect the motion of the "ether" yielded a null result.

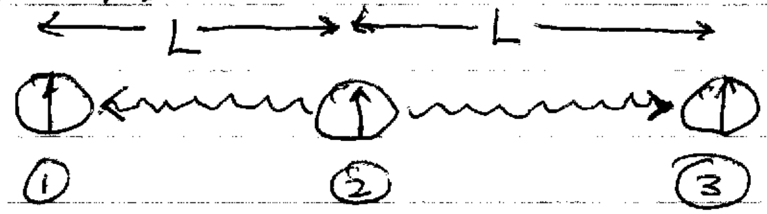
Einstein formulated the two postulates of special relativity,

1) <u>first postulate</u>
The laws of nature and the results of all experiments are independent of the translational motion of the system.
<u>second postulate</u>
2) The speed of light is independent of the motion of the source.

We want to show that the second postulate can be used to derive the Lorentz transformation. This requires observers in different reference frames to measure events with a set of clocks. This requires that the clocks be <u>synchronized</u>.

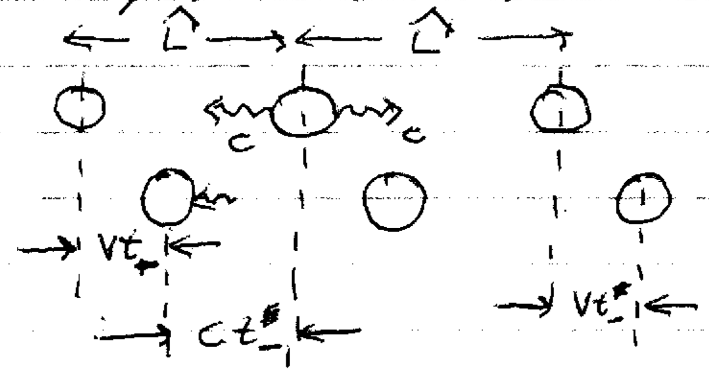
Synchronization of clocks

Suppose we have 3 observers in a stationary system S, each a distance L apart.



Observer ② sends a signal out at $t = 0$. Tells observers ① and ③ to set their clocks at $t = L/c$ when they receive the signal $\Rightarrow$ the three clocks are synchronized.

Now have a group of three observers in a system S' moving with velocity V. repeat this process. The observers in S' claim that their clocks are synchronized. What do the observers S see as the S' system sets their clocks?



with $\hat{L}$ the clock separation in S' seen by S.

What do the observers in S' see?

$\Rightarrow$ the light intersects the left clock at

$$t_- = \frac{L}{c+v}$$

and the light intersect the right clock at

$$t_+ = \frac{L}{c-v}$$

Thus, the events are not simultaneous according to S' and they say that the S' clocks are not properly synchronized.

$\Rightarrow$ in relativity time and space are mixed.

Want to solve for

$$t' = t'(x,t) = \text{time in moving frame}$$

Must have a linear relation between t' and t and x.

$$t' = at + bx \Rightarrow \text{solve for } a, b$$

$\Rightarrow$ for a fixed time t, a shift in t' for a given shift in x must be independent of x

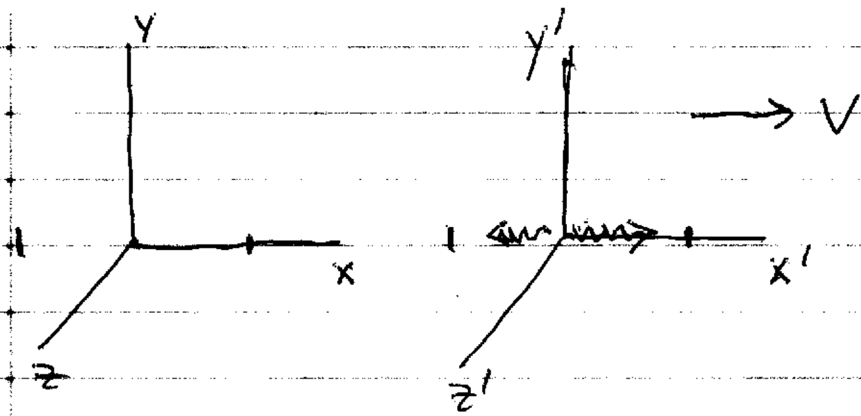
$\Rightarrow$ same for a fixed x

$\Rightarrow$ space and time must be homogeneous

$\Rightarrow$ only differences matter

Consider a reference frame S at rest and S' moving as before. Each has clocks set a distance L apart in their own frames. Let $x = x'$ at $t = t' = 0$.

$\Rightarrow$ origins coincide at $t = t' = 0$

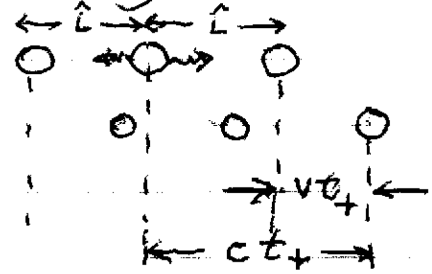


Each sends a pulse out at $t = t' = 0$.
Each says the pulse arrives at L/c.

What does S see when observing S'?
S sees separation of clocks in S' as some $\hat{L}$. The pulse hits the right clock after a time t_+ given by

$$c t_+ = \hat{L} + v t_+$$

since the right clock is moving with velocity V. Thus,



$$t_+ = \frac{\hat{L}}{c-V}$$

At this time $t' = \frac{L}{c}$ ~~so~~ while

$$t = t_+ = \frac{\hat{L}}{c-V}$$

At this time

$$X = c t_+ = \frac{c}{c-V}\hat{L}$$

Inserting into $t' = at + bX$

$$\frac{L}{c} = a\frac{\hat{L}}{c-V} + b\frac{c}{c-V}\hat{L}$$

The observers in S looking at the left moving signal see everything the same but with $X \to -X$ and $V \to -V$ so

$$\frac{L}{c} = a\frac{\hat{L}}{c+V} - b\frac{c}{c+V}\hat{L}$$

Subtracting the two relations yields

$$0 = a\underbrace{\left(\frac{1}{c-v} - \frac{1}{c+v}\right)}_{\frac{c+v-(c-v)}{c^2-v^2}} + bc\underbrace{\left(\frac{1}{c-v} + \frac{1}{c+v}\right)}_{\frac{c+v+c-v}{c^2-v^2}}$$

$$av + bc^2 = 0$$

$$b = -a\frac{v}{c^2}$$

Therefore,
$$t' = a\left(t - \tfrac{1}{c}\beta x\right).$$

We also have the inverse transformation,

$$t = a\left(t' + \tfrac{1}{c}\beta x'\right)$$

since S' sees S as having velocity $-v$. Meanwhile in the S' frame the right pulse moves

$$x' = L = ct'$$

so the time in the S frame is

$$t = a\left(t' + \tfrac{1}{c}\beta\, ct'\right)$$
$$= at'(1+\beta)$$

The same argument holds in the S' frame so

$$t' = at(1-\beta)$$

and

$$t = a^2 t(1+\beta)(1-\beta)$$

$$1 = a^2(1-\beta^2)$$

$$\Rightarrow a = \gamma = \frac{1}{(1-\beta^2)^{1/2}}$$

So,

$$t' = \gamma\left(t - \frac{\beta}{c}x\right)$$

$$t = \gamma\left(t' + \frac{\beta}{c}x'\right)$$

Solve for $x'(x,t) \Rightarrow$ eliminate t'

$$t = \gamma\left[\gamma\left(t - \frac{\beta}{c}x\right) + \frac{\beta}{c}x'\right]$$

$$\frac{\beta}{c}\gamma^2 x + t\underbrace{\underbrace{(1-\gamma^2)}_{1-\frac{1}{1-\beta^2}}}_{-\beta^2\gamma^2} = \gamma\frac{\beta}{c}x'$$

$$\frac{\beta}{c}\gamma^2 x - \beta^2\gamma^2 t = \gamma\frac{\beta}{c}x'$$

$$\boxed{\begin{aligned} x' &= \gamma\,(x - \beta c t) \\ c t' &= \gamma\,(c t - \beta x) \end{aligned}}$$

Lorentz Transformation

Can also show

$$y' = y$$
$$z' = z$$

Four vectors

The Lorentz transformation transforms space and time from one coordinate system to another

$\Rightarrow$ similar to a rotation in 3D

In 3D have a vector $\underset{\sim}{x}$ with components x_1, x_2, x_3. In ~~space~~-time we speak of 4-vectors with components,

$$\left.\begin{aligned} x_0 &= ct \\ x_1 &= x \\ x_2 &= y \\ x_3 &= z \end{aligned}\right\} \qquad \begin{aligned} x_0' &= \gamma(x_0 - \underset{\sim}{\beta}\cdot\underset{\sim}{x}) \\ x_\parallel' &= \gamma(x_\parallel - \beta x_0) \\ x_\perp' &= x_\perp \end{aligned}$$

More generally we will find that the Lorentz transformation also describes other 4-vectors so we will write a general 4-vector A as

$$\left.\begin{aligned} (A_0, A_1, A_2, A_3) \\ (A_0, \underset{\sim}{A}) \end{aligned}\right\} \qquad \begin{aligned} A_0' = \gamma(A_0 - \underset{\sim}{\beta}\cdot\underset{\sim}{A}) \\ \text{etc.} \end{aligned}$$

It is also useful to define a scalar product that is invariant under Lorentz Transformations (LT) just as the dot product is invariant under rotations.

Define

$$A^\mu = (A_0, \underset{\sim}{A}) \quad \text{contravariant}$$

$$A_\mu = (A_0, -\underset{\sim}{A}) \quad \text{covariant}$$

$$A \cdot B = A_0 B_0 - \underset{\sim}{A} \cdot \underset{\sim}{B} = A_\mu B^\mu$$

where the indices are summed. That this product is invariant under LT is easily checked,

$$\underset{\sim}{A}' \cdot \underset{\sim}{B}' = A_0' B_0' - \underset{\sim}{A}' \cdot \underset{\sim}{B}'$$

$$= A_0' B_0' - A_{\parallel}' B_{\parallel}' - A_{\perp}' \cdot B_{\perp}'$$

$$= \gamma^2 (A_0 - \beta A_{\parallel})(B_0 - \beta B_{\parallel})$$
$$- \gamma^2 (A_{\parallel} - \beta A_0)(B_{\parallel} = \beta B_0) - \underset{\sim}{A}_{\perp} \cdot \underset{\sim}{B}_{\perp}$$

$$= \gamma^2 \left[A_0 B_0 (1 - \beta^2) - A_{\parallel} B_{\parallel} (1 - \beta^2) \right] - \underset{\sim}{A}_{\perp} \cdot \underset{\sim}{B}_{\perp}$$

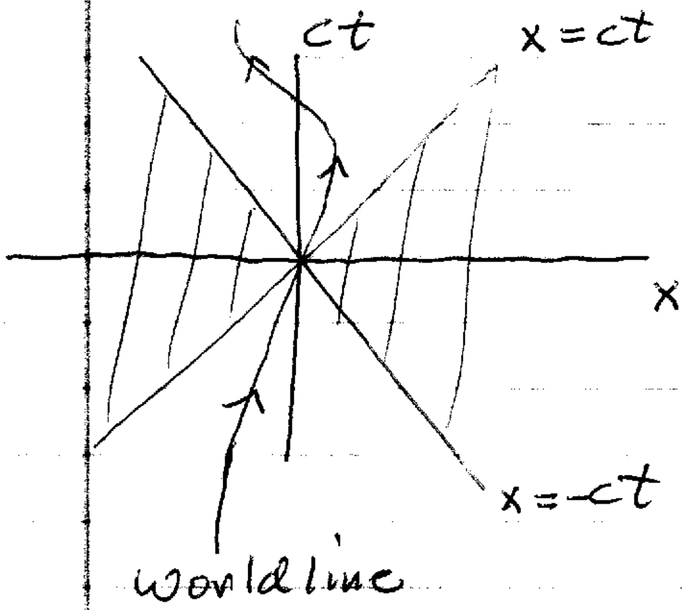
$$= A_0 B_0 - \underset{\sim}{A} \cdot \underset{\sim}{B} = A \cdot B$$

Thus, the 4-vector scalar product can be evaluated in any reference frame.

$\Longrightarrow$ very useful tool!

Classifications of separations in space-time

Events taking place can be ~~separa~~ described in space-time as causally or non-causally connected. This can be understood from the light cone,



Consider a particle that is at $x = 0$ at $t = 0$.

Space-time is separated into three regions by the surface

$$x^2 + y^2 + z^2 = c^2 t^2$$

which defines a light cone. The velocity of any particle is less than c so the world line does not cross into the cross-hatched region. $t > 0$ is the future and $t < 0$ is the past.

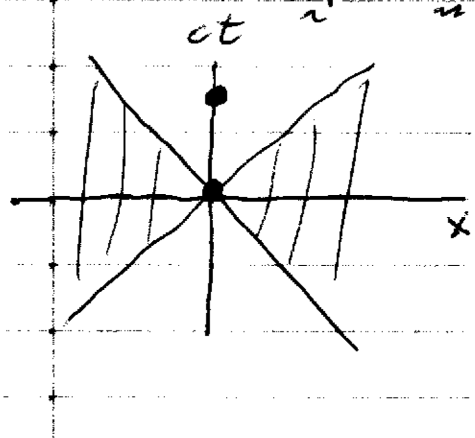
Consider the separation between two events in space-time: $(ct_1, \underset{\sim}{x}_1)$ and $(ct_2, \underset{\sim}{x}_2)$. We can define the separation s_{12} as

$$s_{12}^2 = (\underset{\sim}{x}_1 - \underset{\sim}{x}_2)\cdot(\underset{\sim}{x}_1 - \underset{\sim}{x}_2)$$
$$= c^2(t_1 - t_2)^2 - |\underset{\sim}{x}_1 - \underset{\sim}{x}_2|^2$$

The separation can be classified as

$$\begin{array}{ll} s_{12}^2 > 0 & \text{time-like} \\ \quad < 0 & \text{space-like} \\ \quad = 0 & \text{light-like} \end{array}$$

For $s_{12}^2 > 0$, can choose a Lorentz Transformation to a new frame S' with $\underset{\sim}{x}_1' = \underset{\sim}{x}_2'$



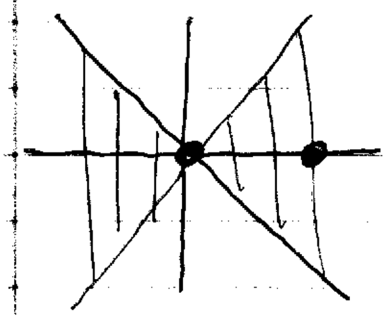
$$s_{12}^2 = c^2(t_1' - t_2')^2 > 0$$

$\Rightarrow$ the particles are at the same position but at different times

For $s_{12}^2 < 0$, can transform to a frame in which $t_1' = t_2'$ for which

$$s_{12}^2 = -|x_1' - x_2'|^2 < 0 .$$

The events occur at the same time but at different spatial locations



Two events with a spacelike separation can not be causally connected.

Proper time

Consider a particle moving with a velocity $\underline{u}(t)$. During a time interval dt, the particle moves a distance

$$d\underline{x} = \underline{u}\, dt$$

The invariant interval ds is

$$\begin{aligned} ds^2 &= c^2 dt^2 - |d\underline{x}|^2 \\ &= c^2 dt^2 - u^2 dt^2 \\ &= c^2 dt^2 (1 - \beta^2) \end{aligned}$$

In the instantaneous rest frame of the particle

$$dt' = d\tau \,, \quad d\underset{\sim}{x}' = 0$$

so

$$ds^2 = c^2 d\tau^2$$

and

$$d\tau = dt \left(1-\beta^2\right)^{1/2} = \frac{dt}{\gamma}$$

$$d\tau \equiv \frac{dt}{\gamma}$$

where τ is the proper time (time in the rest frame of an object).

Time dilation

Since $\gamma \geq 1$,

$$dt = \gamma \, d\tau > d\tau$$

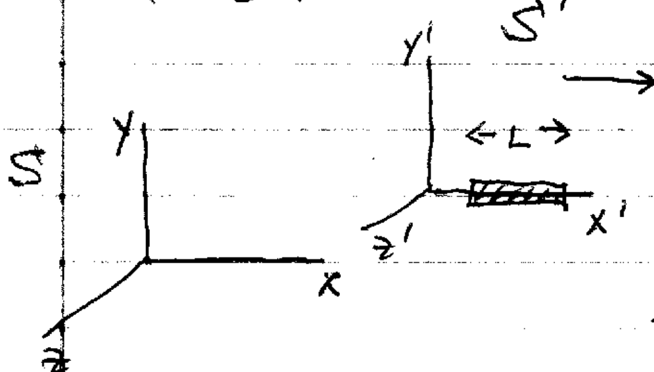
In a frame where the clock is moving time runs faster than in the rest frame of the clock.

One of the first confirmations of special relativity concerned the decay of high velocity particles. Particles with velocities close to the speed of light in air showers were observed to travel further than expected based on their lifetimes. If the lifetime of a particle in its rest frame is τ_0, then the lifetime in the lab frame is

$$t_{lab} = \gamma \tau_0 > \tau_0$$

Lorentz contraction

Suppose a stick of length L in its rest frame is accelerated to high velocity $\underset{\sim}{V}$. What is its length as seen in the lab frame?



$$x' = \gamma(x - \beta c t)$$

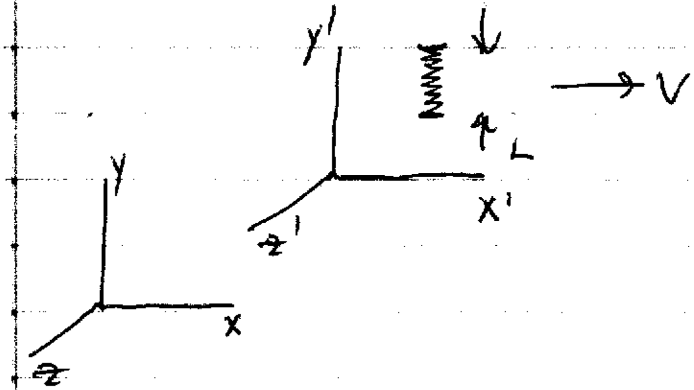
The important point is that the lab. observes measure the two endpoints of the stick at the same time in their frame

$$\Rightarrow \quad L = x_1' - x_2' = \gamma(x_1 - x_2)$$

with $t_1 = t_2$

$$\Rightarrow \quad x_1 - x_2 = \frac{L}{\gamma} < L$$

$\Rightarrow$ The S observers see the stick as shortened.



$$L = y_1' - y_2' = y_1 - y_2$$

$\Rightarrow$ no shortening transverse to motion

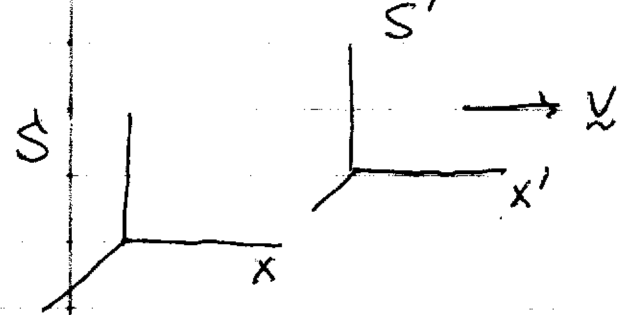
Addition of velocities

In a Galilean transformation we simply add the velocity of an object to the frame velocity. Suppose in a frame S an object is moving with velocity $\underset{\sim}{u}$. In a coordinate system S' moving with velocity $\underset{\sim}{V}$ with respect to S the velocity is

$$\underset{\sim}{u}' = \underset{\sim}{u} - \underset{\sim}{V}$$

This is not correct when $\underset{\sim}{u}$ and $\underset{\sim}{V}$ are close to c since could have $u' \sim 2c$, which is not possible.

Consider a system S' moving along the x direction with respect to S.



Given that the velocity in S is $\underset{\sim}{u}$. What is the velocity $\underset{\sim}{u}'$ in S'?

$$ct' = \gamma_v(ct - \beta_v x)$$
$$x' = \gamma_v(x - \beta_v ct)$$
$$y' = y$$
$$z' = z$$

$$\gamma_v = \frac{1}{(1-\beta_v^2)^{1/2}}$$

$$\beta_v = \frac{V}{c}$$

$$u_x' = \frac{dx'}{dt'} = \frac{\gamma_v(dx - \beta_v c\, dt)}{\gamma_v(dt - \frac{\beta}{c} dx)}$$

$$= \frac{u_x - v}{1 - \frac{\beta_v}{c} u_x}$$

$$\boxed{u_x' = \frac{u_x - v}{1 - \frac{v u_x}{c^2}}}$$

$$u_y' = \frac{dy'}{dt'} = \frac{dy}{\gamma_v(dt - \beta_v \frac{1}{c} dx)} = \frac{u_y}{\gamma_v(1 - \frac{v u_x}{c^2})}$$

$$\boxed{u_y' = \frac{u_y}{\gamma_v(1 - \frac{v u_x}{c^2})} \qquad u_z' = \frac{u_z}{\gamma_v(1 - \frac{v u_x}{c^2})}}$$

For u along x,

$$u' = \frac{u - v}{1 - \frac{vu}{c^2}}$$

For $\frac{vu}{c^2} \ll 1$ Galilean result

For $u \sim -v \sim c$, $u' \simeq \frac{c + c}{1+1} \sim c$

$\Rightarrow u'$ is always smaller than c.

Note that $\underset{\sim}{u}$ does not transform like a 4-vector

$\Rightarrow$ transverse components change under frame shift

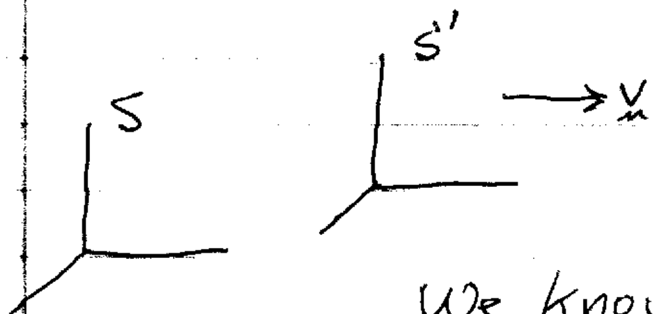
$\Rightarrow$ What is the zero component?

Momentum and Newton's Law

Suppose we have a reference frame S' in which the velocity $\underset{\sim}{u}'$ of an object is much smaller than c. Then from Newton's law we have

$$m \frac{d\underset{\sim}{u}'}{dt} = \underset{\sim}{F}' = \text{force}$$

What is the force law in a reference frame S where $v \lesssim c$?



$$ct' = \gamma(ct - \beta x)$$
$$x' = \gamma(x - \beta ct)$$

We know

$$u_{\parallel}' = \frac{u_{\parallel} - v}{1 - \underset{\sim}{\beta}\cdot\underset{\sim}{u}\frac{1}{c}}$$

$$\underset{\sim}{u}_{\perp}' = \frac{\underset{\sim}{u}_{\perp}}{\gamma(1 - \underset{\sim}{\beta}\cdot\underset{\sim}{u}\frac{1}{c})}$$

$\beta = \beta(v)$
$\gamma = \gamma(v)$

with $\parallel$ along $\underline{v}$ and $\perp$ perpendicular to $\underline{v}$.

Consider a small change in $u_\parallel' \Rightarrow du_\parallel'$ related to a change in $u_\parallel$.

$$du_\parallel' = \frac{du_\parallel}{1-\frac{\beta u_\parallel}{c}} + \frac{(u_\parallel - v)}{\left(1-\frac{\beta u_\parallel}{c}\right)^2}\frac{v}{c^2}du_\parallel$$

$$= \frac{du_\parallel}{\left(1-\frac{\beta u_\parallel}{c}\right)^2}\left[1 - \cancel{\frac{\beta u_\parallel}{c}} + \frac{v}{c^2}(\cancel{u_\parallel} - v)\right]$$

$$= \frac{du_\parallel}{\left(1-\frac{\beta u_\parallel}{c}\right)^2}(1-\beta^2)$$

$$\simeq \frac{du_\parallel}{1-\beta^2}$$

since $u_\parallel \sim v$ when the correction in the denominator is significant.

The time increment dt' is given by

$$c\,dt' = \gamma(c\,dt - \beta\,dx)$$

$$= \gamma c\,dt\left(1-\frac{\beta u_\parallel}{c}\right)$$

$$\simeq \gamma c\,dt\,(1-\beta^2)$$

$$\frac{du_{\parallel}'}{dt'} = \frac{du_{\parallel}}{(1-\beta^2)} \frac{1}{\gamma\, dt\,(1-\beta^2)}$$

$$= \frac{du_{\parallel}}{dt} \frac{1}{(1-\beta^2)^{3/2}}$$

Consider

$$\frac{d}{dt} \frac{u_{\parallel}}{\left(1-\frac{u_{\parallel}^2}{c^2}\right)^{1/2}} = \frac{1}{(\;)^{1/2}} \frac{du_{\parallel}}{dt} + \frac{1}{2} \frac{u_{\parallel}}{(\;)^{3/2}} \frac{2u_{\parallel}}{c^2}\frac{du_{\parallel}}{dt}$$

$$= \frac{1}{\left(1-\frac{u_{\parallel}^2}{c^2}\right)^{3/2}} \frac{du_{\parallel}}{dt} \left[1 - \frac{u_{\parallel}^2}{c^2} + \frac{u_{\parallel}^2}{c^2}\right]$$

$$= \frac{1}{\left(1-\frac{u_{\parallel}^2}{c^2}\right)^{3/2}} \frac{du_{\parallel}}{dt} = \frac{1}{(1-\beta^2)^{3/2}} \frac{du_{\parallel}}{dt}$$

Thus, in the ~~lab~~ S' frame

$$\boxed{m \frac{d}{dt} \gamma u_{\parallel} = F_{\parallel}}$$

with $F_{\parallel}$ the force in the S' frame (to be calculated) and

$$\gamma = \frac{1}{(1-\beta^2)^{1/2}} = \frac{1}{\left(1-\frac{u_{\parallel}^2}{c^2}\right)^{1/2}}$$

$m\gamma u_{\parallel} \equiv P_{\parallel}$ = relativistic momentum

Similarly for $\underset{\sim}{u}_\perp'$

$$d\underset{\sim}{u}_\perp' = \frac{d\underset{\sim}{u}_\perp}{\gamma\left(1-\beta u_{\parallel}\frac{1}{c}\right)} + \frac{\underset{\sim}{u}_\perp \frac{\beta}{c} du_{\parallel}}{\gamma\left(1-\beta u_{\parallel}\frac{1}{c}\right)^2}$$

and

$$\frac{d\underset{\sim}{u}_\perp'}{dt'} = \frac{d\underset{\sim}{u}_\perp}{dt}\left[\frac{1}{\gamma}\frac{1}{(1-\beta^2)}\ \frac{1}{\gamma}\frac{1}{(1-\beta^2)}\right]$$

$$+ \frac{\underset{\sim}{u}_\perp \frac{\beta}{c}\frac{du_{\parallel}}{dt}}{\gamma^2(1-\beta^2)^3}$$

$$= \frac{1}{1-\beta^2}\frac{d\underset{\sim}{u}_\perp}{dt} + \frac{\beta}{c}\frac{du_{\parallel}}{dt}\frac{1}{(1-\beta^2)^2}\underset{\sim}{u}_\perp$$

$$\frac{d}{dt}\frac{1}{(1-\beta^2)^{1/2}} = -\frac{1}{2}\frac{1}{(\ )^{3/2}}\left(-2\beta\frac{1}{c}\frac{du_{\parallel}}{dt}\right)$$

$$= \frac{\beta}{c}\frac{du_{\parallel}}{dt}\frac{1}{(1-\beta^2)^{3/2}}$$

and

$$\frac{d\underset{\sim}{u}_\perp'}{dt'} = \frac{1}{1-\beta^2}\frac{d\underset{\sim}{u}_\perp}{dt} + \frac{\underset{\sim}{u}_\perp \frac{d}{dt}\gamma}{(1-\beta^2)^{1/2}}$$

$$\frac{d\underset{\sim}{u}_\perp'}{dt} = \gamma\left(\frac{d}{dt}(\gamma \underset{\sim}{u}_\perp)\right)$$

$$m \frac{d}{dt}\left(\gamma \underset{\sim}{u}_\perp\right) = \frac{1}{\gamma}\underset{\sim}{F}_\perp' = \underset{\sim}{F}_\perp$$

Define the total momentum

$$\underset{\sim}{p} = m\gamma \underset{\sim}{u}$$

and the force law in the S' frame is

$$\frac{d}{dt}\underset{\sim}{p} = \underset{\sim}{F} \quad \text{with } \gamma = \frac{1}{\left(1-\frac{u^2}{c^2}\right)^{1/2}}$$

Kinetic Energy

In the non-relativistic limit the kinetic energy of an object with velocity $\underset{\sim}{u}$ is

$$T(u) = \frac{1}{2} m u^2$$

What is the relativistic generalization? From the force eqn,

$$\frac{d}{dt} m\gamma(u)\underset{\sim}{u} = \underset{\sim}{F}$$

The rate at which work is done on an object is

$$\underset{\sim}{F} \cdot \underset{\sim}{u} = \underset{\sim}{u} \cdot \frac{d}{dt} m \gamma(u) \underset{\sim}{u}$$

$$= m\gamma \frac{d}{dt} \frac{u^2}{2} + m u^2 \frac{d}{dt} \gamma$$

$$= m\gamma \frac{d}{dt} \frac{u^2}{2} + m u^2 \left(-\frac{1}{2} \frac{1}{(1-\beta^2)^{3/2}} \left(-\frac{1}{c^2} \frac{d}{dt} u^2 \right) \right)$$

$$= m \left(\frac{d}{dt} \frac{u^2}{2} \right) \left[\frac{1}{(1-\beta^2)^{1/2}} + \frac{\beta^2}{(1-\beta^2)^{3/2}} \right]$$

$$= \frac{mc^2}{(1-\beta^2)^{3/2}} \frac{d}{dt} \left(\frac{u^2}{2c^2} \right) = mc^2 \frac{d}{dt} \frac{1}{(1-\beta^2)^{1/2}}$$

$$= \frac{d}{dt} \left(\gamma m c^2 \right)$$

But the ~~work~~ rate at which work is done is the rate of change of energy so

$$W \equiv \gamma m c^2 = \text{energy of object}$$

$$\frac{d}{dt} W = \underset{\sim}{F} \cdot \underset{\sim}{u}$$

$$\Rightarrow \text{kinetic energy} \equiv T = \gamma m c^2 - m c^2$$

In the non-relativistic limit

$$T = \frac{m_0 c^2}{(1-\beta^2)^{1/2}} - mc^2 \simeq mc^2\left(1 + \tfrac{1}{2}\beta^2 - 1\right)$$
$$= \tfrac{1}{2} m u^2$$

$\Rightarrow$ as expected

Energy/momentum 4-vector

How do we construct a 4-vector that represents velocity? We know that

$$(ct, \underset{\sim}{x})$$

is a 4-vector and so is

$$(c\,dt, d\underset{\sim}{x})$$

We can divide this by a Lorentz invariant $d\tau$ and we will also have a 4-vector

$$\left(c\frac{dt}{d\tau}, \frac{d\underset{\sim}{x}}{d\tau}\right)$$

For a particle with velocity $\underset{\sim}{u}$,

$$d\tau = \frac{dt}{\gamma}$$

so that

$$\left(\gamma c \frac{dt}{dt}, \gamma \frac{d}{dt} \underset{\sim}{x}\right)$$

or

$$(\gamma c, \gamma u)$$

is a 4-vector. We can also multiply this by any Lorentz invariant. The mass of a particle in its rest frame is an invariant so

$$(m\gamma c, m\gamma \underset{\sim}{u})$$

is a 4-vector and we have the energy momentum 4-vector

$$\left(\frac{E}{c}, \underset{\sim}{P}\right) = (P_0, \underset{\sim}{P})$$

with $P_0 = E/c$. The energy is the time component of the momentum 4-vector. The length of a 4-vector is a Lorentz invariant so

$$P \cdot P = \frac{E^2}{c^2} - \underset{\sim}{P} \cdot \underset{\sim}{P} = m^2 c^2$$

where we have evaluated $\underset{\sim}{P} \cdot \underset{\sim}{P}$ in the particle rest frame. Thus,

$$\boxed{E^2 = p^2c^2 + m^2c^4}$$

This expresses the particle energy in terms of its momentum $\underset{\sim}{p}$.

The momentum and energy clearly depend on the reference frame of the observer. The mass m is the only invariant of the Energy / momentum 4-vector. We can also express the velocity of a particle in terms of its momentum and energy,

$$\underset{\sim}{u} = \frac{\underset{\sim}{p}}{\gamma m} = \frac{\underset{\sim}{p}\, c^2}{\gamma m c^2} = \frac{\underset{\sim}{p}\, c^2}{E}$$

For $E \gg mc^2$, $E \gtrsim pc$ so

$$\underset{\sim}{u} \lesssim \frac{\underset{\sim}{p}}{p}\, c$$

For photons, $E = pc$ and $u = c$.

Relativistic Kinematics

In investigating collisions between particles or objects in Newtonian mechanics, we use laws for the conservation of the total momentum and total energy before and after the collision,

$$\underset{\sim}{P}_{tot} = const = \sum_i \underset{\sim}{P}_i$$

$$E_{TOT} = const = \sum_i T_i$$

or

$$E_{tot} = \sum_i T_i + \sum_i m_i c^2$$

In special relativity we write the conservation of energy and momentum as the 4-vector equation

$$\sum_i p^{\mu} = const.$$

This includes the three components of the vector momentum and the energy. The invariance of the scalar product under Lorentz transformations enables you to very simply investigate collisions between charged particles.

Example

Consider two particles 1 and 2 with momenta P_1^{μ}, P_2^{μ}

$$P_1 \cdot P_2 = \frac{E_1 E_2}{c^2} - P_1$$

~~$P_2 \cdot P_1$~~ ~~$P_1 \cdot P_2$~~

In the reference frame of particle # 1 this is

$$P_1 \cdot P_2 = \frac{m_1 c^2}{c^2} E_2 \quad \text{etc.}$$

You can use the invariance in doing the hwk.