Green's function for the wave equation

We want to begin exploring wave generation resulting from oscillating currents and charges. To do this we need to calculate the Green's function response of a wave equation to a specified source. This will allow us to explore how Maxwell's eqns respond to oscillatory sources.
We have the generic form

$$\nabla^2 \psi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi = -4\pi\, S(\underset{\sim}{x},t)$$

To solve this equation for $\psi(\underset{\sim}{x},t)$ we need the Green's function

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\underset{\sim}{x},\underset{\sim}{x}',t,t') = -4\pi\,\delta(\underset{\sim}{x}-\underset{\sim}{x}') \otimes \delta(t-t')$$

where $G = 0$ for $t \leq t'$. Namely, G is the response function source that turns on at $t = t'$. Take the Fourier transform of the equation in space and the Laplace transform in time,

$$\hat{G}(\underset{\sim}{k},\omega) = \int_{-\infty}^{\infty} d\underset{\sim}{x} \int_{0}^{\infty} dt\; e^{-i\underset{\sim}{k}\cdot\underset{\sim}{x}}\, e^{i\omega t}\, G(\underset{\sim}{x},t)$$

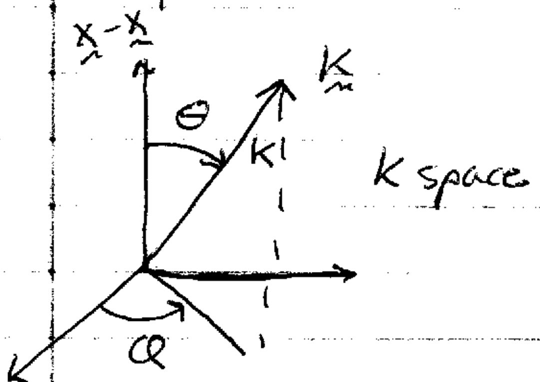
Operating on the wave equation,

$$\left(-k^2+\frac{\omega^2}{c^2}\right)\hat{G} = -4\pi\, e^{-i\underset{\sim}{k}\cdot\underset{\sim}{x}'}\, e^{i\omega t'}$$

~~whate~~ where $G = 0$ for $t = 0 < t'$.

$$\hat{G} = \frac{4\pi\, e^{i\omega t'}\, e^{-i\underset{\sim}{k}\cdot\underset{\sim}{x}'}}{k^2-\frac{\omega^2}{c^2}}$$

$$G(\underset{\sim}{x},t) = 4\pi \int_{-\infty}^{\infty} d\underset{\sim}{k} \int_C d\omega \frac{1}{(2\pi)^4}\, \frac{e^{-i\omega(t-t')}\, e^{i\underset{\sim}{k}\cdot(\underset{\sim}{x}-\underset{\sim}{x}')}}{k^2-\frac{\omega^2}{c^2}}$$

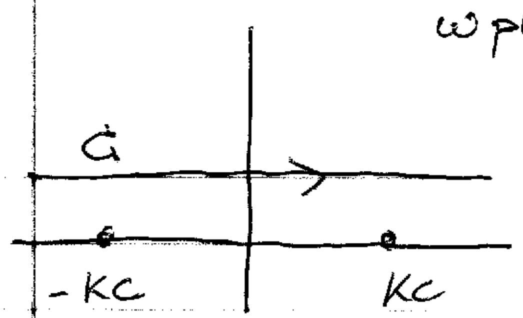


$$d\underset{\sim}{k} = d(\cos\theta)\, d\alpha\, k^2 dk$$

$$G = \frac{4\pi(2\pi)}{(2\pi)^4} \int_C d\omega\, e^{-i\omega(t-t')} \int_0^{\infty} dk\, k^2 \otimes \int_{-1}^{1} d(\cos\theta)\, e^{ik|\underset{\sim}{x}-\underset{\sim}{x}'|\cos\theta}$$

$$= \frac{1}{2\pi^2} \int_0^{\infty} dk\, k^2 \int_C d\omega\, \frac{e^{-i\omega(t-t')}}{k^2-\frac{\omega^2}{c^2}} \left(\frac{e^{ik|\underset{\sim}{x}-\underset{\sim}{x}'|} - e^{-ik|\underset{\sim}{x}-\underset{\sim}{x}'|}}{ik|\underset{\sim}{x}-\underset{\sim}{x}'|}\right)$$

Require $G = 0$ for $t < t'$.

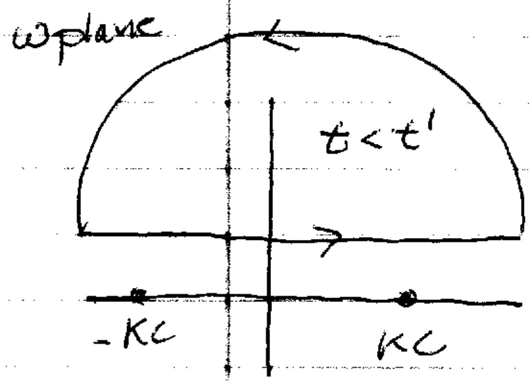


C must lie above the singularities at $\pm kc$

For $t < t'$, $e^{-i\omega(t-t')} = e^{i\omega(t'-t)}$ can close the contour in the UHP where

$$e^{i\omega(t'-t)} \to 0$$

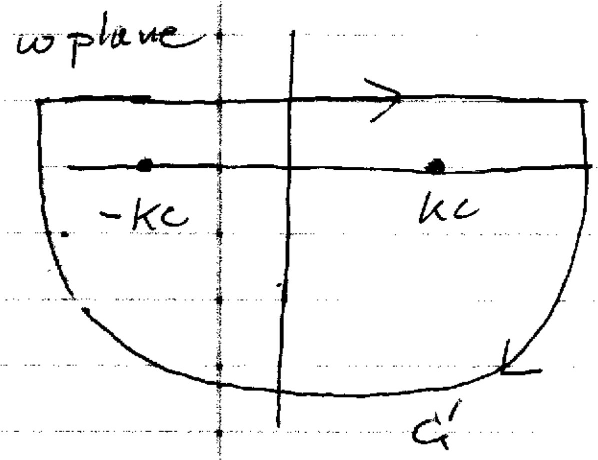
since $\mathrm{Im}(\omega) > 0$ and $t' > t$.



From Jordan's lemma can discard contribution from circle $\Rightarrow$ integrand $\sim \frac{1}{\omega^2}$.

$\Rightarrow$ no singularities in UHP so $G = 0$ for $t' > t$.

For $t > t'$, can close the contour C in the LHP and evaluate the residues at $\omega = \pm kc$.



$$G = \frac{-c^2}{2\pi^2} \int_0^\infty dk\, k \left(\frac{e^{ik|\underset{\sim}{x}-\underset{\sim}{x}'|} - e^{-ik|\underset{\sim}{x}-\underset{\sim}{x}'|}}{i|\underset{\sim}{x}-\underset{\sim}{x}'|} \right)$$

$$\otimes \underbrace{\int_{C'} d\omega \frac{e^{-i\omega(t-t')}}{(\omega-kc)(\omega+kc)}}$$

$$-2\pi i \left(\frac{e^{-ikc(t-t')}}{2kc} + \frac{e^{ikc(t-t')}}{-2kc} \right)$$

$$G = \frac{c}{\pi} \int_0^\infty dk \, \frac{\sin(k|\underset{\sim}{x}-\underset{\sim}{x}'|)}{|\underset{\sim}{x}-\underset{\sim}{x}'|} \underbrace{\sin[kc(t-t')]}_{\mathrm{Im}\, e^{ikc(t-t')}}$$

$$= \frac{c}{\pi} \frac{1}{|\underset{\sim}{x}-\underset{\sim}{x}'|} \mathrm{Im} \int_{-\infty}^{\infty} \frac{dkc}{c} e^{ikc(t-t')} \left(\frac{e^{ik|\underset{\sim}{x}-\underset{\sim}{x}'|} - e^{-ik|\underset{\sim}{x}-\underset{\sim}{x}'|}}{2i} \right)$$

$$= \frac{1}{|\underset{\sim}{x}-\underset{\sim}{x}'|} \mathrm{Im}(-i) \Big[\cancel{\delta\left[\frac{|\underset{\sim}{x}-\underset{\sim}{x}'|}{c} + t - t' \right]}^{\,0}$$
$$- \delta\left[\frac{|\underset{\sim}{x}-\underset{\sim}{x}'|}{c} - t + t' \right] \Big]$$

The argument in the first δ-function is positive since $t > t'$ so is discarded

$$G = \frac{1}{|\underset{\sim}{x}-\underset{\sim}{x}'|} \delta\left[t' - \left(t - \frac{|\underset{\sim}{x}-\underset{\sim}{x}'|}{c} \right) \right]$$

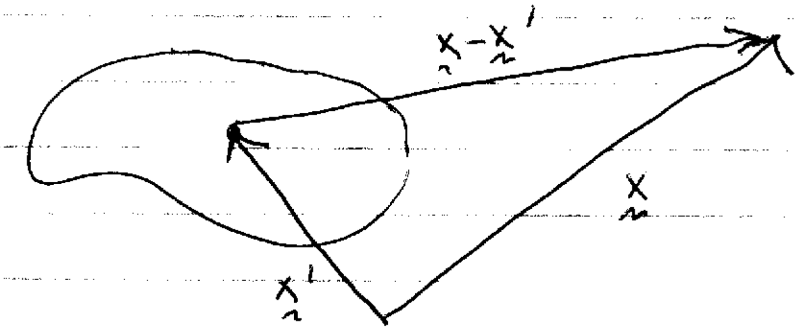
This is the retarded Green's function. At the position $|\underset{\sim}{x}-\underset{\sim}{x}'|$ the source is evaluated at the earlier time

$$t - \frac{|\underset{\sim}{x}-\underset{\sim}{x}'|}{c} \Rightarrow \text{retarded time}$$

due to the fact that the signal takes a finite time to propagate a distance $|\underset{\sim}{x}-\underset{\sim}{x}'|$ from the source.

Thus, the solution $\psi(x,t)$ is given by

$$\psi = \int d\underset{\sim}{x}'\,dt' \frac{S(\underset{\sim}{x}',t')}{|\underset{\sim}{x}-\underset{\sim}{x}'|}\,\delta\left[t' - \left(t - \frac{|\underset{\sim}{x}-\underset{\sim}{x}'|}{c}\right)\right]$$

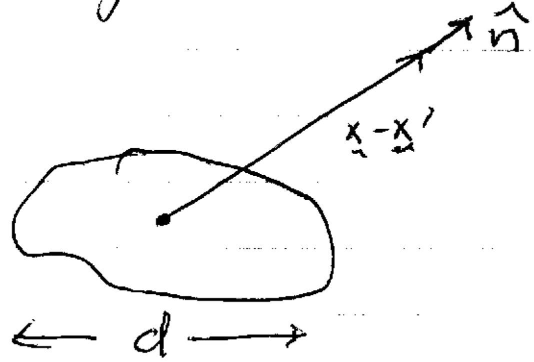


At the point $\underset{\sim}{x}$, we observe the effect of S from an earlier time $t - |\underset{\sim}{x}-\underset{\sim}{x}'|/c$. This accounts for the delay in the propagation of a wave of velocity c from $\underset{\sim}{x}'$ to $\underset{\sim}{x}$. At $t' = t - |\underset{\sim}{x}-\underset{\sim}{x}'|/c$ the source emits a signal that arrives at $\underset{\sim}{x}$ at t.

Fields from a localized oscillating source

Consider a localized region of currents and charges

$$\rho(\underset{\sim}{x})e^{-i\omega t}$$
$$\underset{\sim}{J}(\underset{\sim}{x})e^{-i\omega t}$$



We want to calculate $\underset{\sim}{E}, \underset{\sim}{B}$ produced by these sources

$\Rightarrow$ assume the Lorenz guage

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\underset{\sim}{A} = -\mu_0 \underset{\sim}{J}$$

with $\nabla \cdot \underset{\sim}{A} + \frac{1}{c^2}\frac{\partial}{\partial t}\Phi = 0$

Thus,

$$\underset{\sim}{A} = \frac{\mu_0}{4\pi}\int d\underset{\sim}{x}'\,dt'\,\frac{\underset{\sim}{J}(\underset{\sim}{x}')e^{-i\omega t'}}{|\underset{\sim}{x}-\underset{\sim}{x}'|}\,\delta\left[t' - \left(t - \frac{|\underset{\sim}{x}-\underset{\sim}{x}'|}{c}\right)\right]$$

$$= \frac{\mu_0}{4\pi}\int dx'\,\frac{\underset{\sim}{J}(\underset{\sim}{x}')}{|\underset{\sim}{x}-\underset{\sim}{x}'|}\,e^{-i\omega\left[t - \frac{|\underset{\sim}{x}-\underset{\sim}{x}'|}{c}\right]}$$

$$= \frac{\mu_0}{4\pi}e^{-i\omega t}\int d\underset{\sim}{x}'\,\frac{\underset{\sim}{J}(\underset{\sim}{x}')}{|\underset{\sim}{x}-\underset{\sim}{x}'|}\,e^{ik|\underset{\sim}{x}-\underset{\sim}{x}'|}$$

with $k \equiv \omega/c$.

Thus, the dependence of $\underset{\sim}{A}$ on $|\underset{\sim}{x}-\underset{\sim}{x}'|$ depends both on the distance $|\underset{\sim}{x}-\underset{\sim}{x}'|$ from the source but $k\,|\underset{\sim}{x}-\underset{\sim}{x}'|$. That is, there is a new scale

$$|\underset{\sim}{x}-\underset{\sim}{x}'| \sim \frac{1}{k}.$$

Outside of the source we can evaluate

$$\underset{\sim}{B} = \nabla \times \underset{\sim}{A}$$

and

$$\nabla \times \underset{\sim}{B} = -\frac{i\omega}{c^2}\underset{\sim}{E}$$

$$\underset{\sim}{E} = ic\frac{1}{k}\nabla \times \underset{\sim}{B}$$

Near zone: $k\,|\underset{\sim}{x}-\underset{\sim}{x}'| \ll 1$ but $|\underset{\sim}{x}-\underset{\sim}{x}'| \gg d$

$$\underset{\sim}{A} = \frac{\mu_0}{4\pi} e^{-i\omega t} \int d\underset{\sim}{x}'\, \underset{\sim}{J}(\underset{\sim}{x}')\, \frac{1}{|\underset{\sim}{x}-\underset{\sim}{x}'|}$$

This is the same as the result from magneto statics

$\Rightarrow$ when light travels the distance $|\underset{\sim}{x}-\underset{\sim}{x}'|$ on a time scale short compared with $1/\omega$, have the quasi-static solution.

$\Rightarrow$ $\underset{\sim}{B}$ falls off as $1/|\underset{\sim}{x}|^2$

Far zone: take $|\underset{\sim}{x}'| \sim d$ but $k|\underset{\sim}{x}| \gg 1$ with $|\underset{\sim}{x}| \gg |\underset{\sim}{x}'|$

$$|\underset{\sim}{x} - \underset{\sim}{x}'| = \left(x^2 - 2\,\underset{\sim}{x}\cdot\underset{\sim}{x}' + x'^2\right)^{1/2}$$

$$\simeq x\left(1 - 2\frac{\hat{n}\cdot\underset{\sim}{x}'}{x}\right)^{1/2}$$

$$\simeq x - \hat{n}\cdot\underset{\sim}{x}' = |\underset{\sim}{x}| - \hat{n}\cdot\underset{\sim}{x}'$$

$$\underset{\sim}{A} = \frac{\mu_0}{4\pi} e^{-i\omega t} \frac{e^{ik|\underset{\sim}{x}|}}{|\underset{\sim}{x}|} \underbrace{\int d\underset{\sim}{x}'\, \underset{\sim}{J}(\underset{\sim}{x}')\, e^{-ik\underset{\sim}{x}'\cdot\hat{n}}}$$

Independent of $|\underset{\sim}{x}|, t$ but depends on the angle of the source compared with the observation location (through $\hat{n}$).

$\Rightarrow$ corresponds to an outward propagating wave with phase

$$\delta = k|\underset{\sim}{x}| - \omega t = k(|\underset{\sim}{x}| - ct)$$

$\Rightarrow$ phase front propagates at

$$|\underset{\sim}{x}| = ct$$

$\Rightarrow$ $\underset{\sim}{E}, \underset{\sim}{B} \sim \frac{1}{|\underset{\sim}{x}|}$, $S \sim \frac{1}{|\underset{\sim}{x}|^2}$, energy flux $\sim |\underset{\sim}{x}|^2 S \sim \text{const.}$

Electric Dipole Fields

In the limit where $\mathbf{J}$ is localized over a distance small compared with $k^{-1} = c/\omega$ the source integral is easily evaluated

$\Rightarrow$ dipole limit

$\Rightarrow$ source small compared with free-space wavelength

For $kd \ll 1$,

$$e^{-i\mathbf{x}'\cdot\hat{n}k} \simeq \left(1 - ik\,\mathbf{x}'\cdot\hat{n} + \cdots\right)$$

$$\mathbf{A} \simeq \frac{\mu_0}{4\pi}\frac{e^{ik|\mathbf{x}|}}{|\mathbf{x}|}e^{-i\omega t}\underbrace{\int d\mathbf{x}'\,\mathbf{J}(\mathbf{x}')}$$

$$\underbrace{\int d\mathbf{x}'\,\mathbf{J}(\mathbf{x}')\cdot\underbrace{\nabla'\mathbf{x}'}_{\text{unit tensor}}}$$

$$\underbrace{-\int d\mathbf{x}'\,\mathbf{x}'\,\nabla'\cdot\mathbf{J}}_{\text{integration by parts}}$$

But

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{J} = 0$$

$$\nabla\cdot\mathbf{J} = i\omega\rho$$

$$\underset{\sim}{A} = \frac{\mu_0}{4\pi} \frac{e^{ik|\underset{\sim}{x}|}}{|\underset{\sim}{x}|} e^{-i\omega t} \int d\underset{\sim}{x}' (-i\omega \rho(\underset{\sim}{x}') x')$$

$$\underset{\sim}{P} \equiv \int d\underset{\sim}{x}' \, \underline{x}' \rho(\underset{\sim}{x}') = \text{electric dipole moment}$$

$$\underset{\sim}{A} = -i\omega \frac{\mu_0}{4\pi} \underset{\sim}{P} \frac{e^{ik|\underset{\sim}{x}|}}{|\underset{\sim}{x}|} e^{-i\omega t}$$

When evaluating $\underset{\sim}{B} = \nabla \times \underset{\sim}{A}$, ∇ acts on $e^{ik|\underset{\sim}{x}|}$ in the "~~far~~ zone"

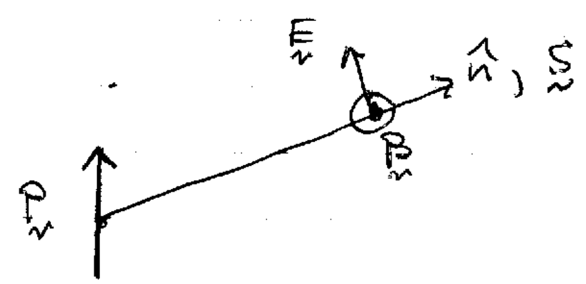
$$\Rightarrow \nabla e^{ik|\underset{\sim}{x}|} = ik \nabla |\underset{\sim}{x}| = ik \hat{n}$$

$$\underset{\sim}{B} = \frac{\mu_0}{4\pi} \omega k \, \hat{n} \times \underset{\sim}{p} \, e^{ik|\underset{\sim}{x}|} e^{-i\omega t} \frac{1}{|\underset{\sim}{x}|}$$

$$= \frac{\mu_0}{4\pi} k^2 c \, \hat{n} \times \underset{\sim}{p} \, e^{ik|\underset{\sim}{x}|} e^{-i\omega t} \frac{1}{|\underset{\sim}{x}|}$$

$\Rightarrow$ falls off as $\frac{1}{|\underset{\sim}{x}|}$

$$\underset{\sim}{E} = i \frac{c}{k} \nabla \times \underset{\sim}{B} = c \, \underset{\sim}{B} \times \hat{n} \sim \frac{1}{|\underset{\sim}{x}|}$$



Power radiated

$$\overline{\vec{S}}\cdot\hat{n} = \frac{1}{2}\frac{1}{\mu_0}\hat{n}\cdot \underset{\sim}{E}_0\times \underset{\sim}{B}_0^*$$

$$= \frac{c}{2\mu_0}\hat{n}\cdot\left((\underset{\sim}{B}_0\times\hat{n})\times B_0^*\right) = \underbrace{\frac{|B_0|^2}{2\mu_0}}_{\bar{u}}c$$

$$= \frac{c}{2\mu_0}\epsilon_0\frac{\mu_0^2}{16\pi^2}k^4c^2\frac{|\hat{n}\times\underset{\sim}{p}|^2}{\epsilon_0}\frac{1}{|\underset{\sim}{x}|^2}$$

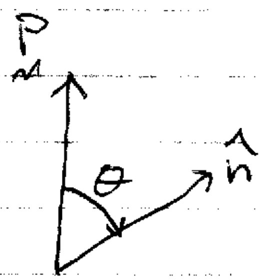
$$= \frac{c}{32\pi^2}k^4|\hat{n}\times\underset{\sim}{p}|^2\frac{1}{\epsilon_0}\frac{1}{|\underset{\sim}{x}|^2}$$

$$\frac{\text{power}}{\text{solid angle}} \equiv |\underset{\sim}{x}|^2\,\overline{\underset{\sim}{S}}\cdot\hat{n} = \frac{dP}{d\Omega}$$

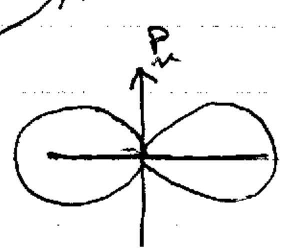
$$\frac{dP}{d\Omega} = \frac{c}{32\pi^2}k^4\frac{1}{\epsilon_0}|\hat{n}\times\underset{\sim}{p}|^2$$

$\Rightarrow$ independent of $|\underset{\sim}{x}|$

$\Rightarrow$ power flux from radiation field is $|\underset{\sim}{x}|$ independent.



$$\frac{dP}{d\Omega} = \frac{c}{32\pi^2}k^4\frac{1}{\epsilon_0}|\underset{\sim}{p}|^2\sin^2\theta$$



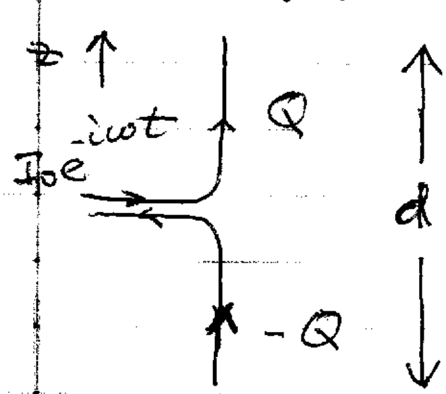
$\Rightarrow$ peaks $\perp$ to $\underset{\sim}{p}$

Integrate over all angles

$$P = \frac{c}{32\pi^2} k^4 |\mathbf{p}|^2 \frac{1}{\epsilon_0} \underbrace{\int_{-1}^{1} d\cos\theta \left(1-\cos^2\theta\right)}_{2 - 2\left(\frac{1}{3}\right) = \frac{4}{3}} 2\pi$$

$$\boxed{P = \frac{c}{12\pi} k^4 |\mathbf{p}|^2 \frac{1}{\epsilon_0}}$$

Center fed antenna:



$$\frac{dQ}{dt} = I_0 e^{-i\omega t}$$

$$Q = \frac{I_0 e^{-i\omega t}}{-i\omega}$$

$$\frac{Q}{d/2} = \frac{2 I_0 e^{-i\omega t}}{-i\omega d} = \frac{\text{charge}}{\text{length}}$$

$\Rightarrow$ $\frac{Q}{d}$ independent of z

$\Rightarrow$ For $kd \ll 1$, charge has time to spread uniformly along antenna

$$p_z = 2 \int_0^{d/2} dz\, z \frac{2 I_0}{-i\omega d} = i \frac{I_0 d}{2\omega}$$

$$P = \frac{c}{12\pi} k^4 \frac{1}{\epsilon_0} \frac{I_0^2 d^2}{4\omega^2} = \frac{1}{48\pi} \sqrt{\frac{\mu_0}{\epsilon_0}} (kd)^2 I_0^2$$

The power increases as $(kd)^2 = \frac{\omega^2 d^2}{c^2}$.

Assumed $kd \ll 1$ in dipole limit. Can calculate radiation for arbitrary kd

$\Rightarrow$ radiation peaks for $kd \sim 1$

Radiation resistance:

The power dissipated in a simple circuit is

$$P = \langle I^2 R \rangle_t = \frac{I_0^2}{2} R$$

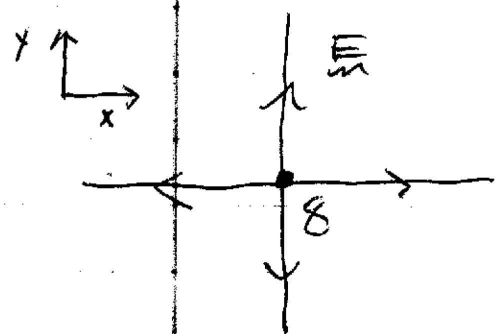
$\Rightarrow$ radiation resistance of antenna

$$R_r \equiv \frac{1}{24\pi} \sqrt{\frac{\mu_0}{\epsilon_0}} (kd)^2$$

$\Rightarrow$ Power is radiated from the antenna so from the point of view of the circuit connected to the antenna there is an effective resistance.

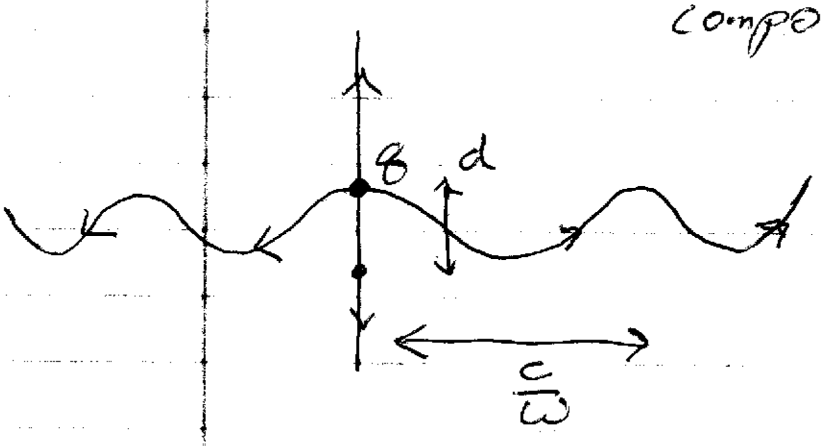
Physical picture of radiation

Consider a charge q. At rest $\underset{\sim}{E}$ points radially outwards. Suppose that q oscillates vertically (y) with frequency ω and amplitude d. Because the propagation velocity of $\underset{\sim}{E}$ at c, the $\underset{\sim}{E}$ takes a sinusoidal form with components of $\underset{\sim}{E}$ along y.



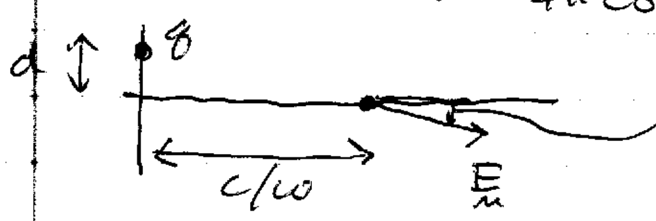
Information about the direction $\underset{\sim}{E}$ propagates outward at velocity c.

$\Rightarrow$ transition from near to far zone at a distance c/ω.



Estimate of $E_t = E_y$:

$$E_t \sim \left(\frac{d}{c/\omega}\right) \frac{q}{4\pi\epsilon_0 \left(\frac{c}{\omega}\right)^2} \sim \frac{p}{4\pi\epsilon_0 (c/\omega)^3}$$



$\theta \sim \dfrac{d}{c/\omega}$

$E_t \sim E \sin\theta$

$E \sim \dfrac{q}{4\pi\epsilon_0 (c/\omega)^2}$

$\sin\theta \sim d/(c/\omega)$

Estimate of B_t:

Vertical oscillation of charge produces B_t in the z direction. From Faraday's law

$$\frac{\partial \underset{\sim}{B}}{\partial t} + \nabla \times \underset{\sim}{E} = 0$$

$$B_t \sim \frac{k}{\omega} E_t \sim \frac{1}{c} E_t$$

$$S \sim E_t H_t \sim E_t \left(\frac{1}{\mu_0}\frac{1}{c} E_t\right) \sim \frac{E_t^2}{\mu_0 c}$$

$$P \sim \underbrace{\left(\frac{E_t^2}{\mu_0 c}\right)}_{\text{Poynting flux}} \underbrace{4\pi \frac{c^2}{\omega^2}}_{\text{area}}$$

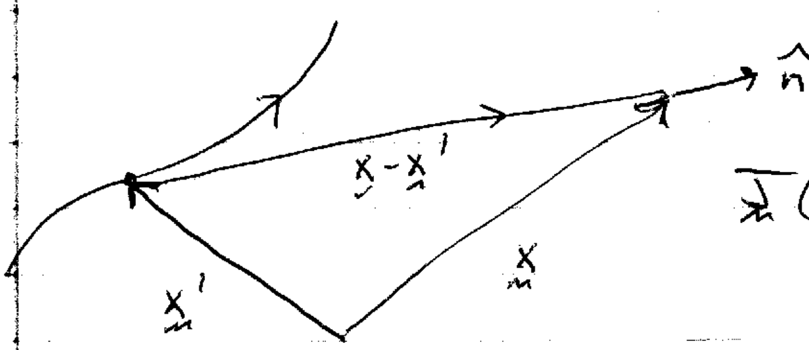
$$P \sim \frac{p^2}{16\pi^2 \epsilon_0^2} \frac{1}{(c/\omega)^6} 4\pi \left(\frac{c}{\omega}\right)^2 \frac{1}{\mu_0 c}$$

$$\sim \frac{k^4 p^2 c}{4\pi \epsilon_0} \quad \text{with } k = \frac{\omega}{c} \text{ and } \mu_0 \epsilon_0 = \frac{1}{c^2}$$

$\Longrightarrow$ same as power radiated in the dipole limit.

Radiation from an accelerating charge

Consider a charge q moving with velocity $\underset{\sim}{v}(t)$ and at a position $\underset{\sim}{r}(t)$



$$\underset{\sim}{J}(\underset{\sim}{x}',t') = q\,\underset{\sim}{v}(t') \otimes \delta[\underset{\sim}{x}'-\underset{\sim}{r}(t')]$$

The vector potential $\underset{\sim}{A}$ associated with the charge is given by the retarded Green's function,

$$\underset{\sim}{A} = \frac{\mu_0}{4\pi}\int dt'\,d\underset{\sim}{x}'\;\frac{\underset{\sim}{J}(\underset{\sim}{x}',t')}{|\underset{\sim}{x}-\underset{\sim}{x}'|}\,\delta\left[t'-\left(t-\frac{|\underset{\sim}{x}-\underset{\sim}{x}'|}{c}\right)\right]$$

$$= \frac{\mu_0}{4\pi}\int dt'\;\frac{q\,\underset{\sim}{v}(t')}{R(t')}\,\delta\left[t'-t+\frac{1}{c}R(t')\right]$$

with $R(t') = |\underset{\sim}{x}-\underset{\sim}{r}(t')| \simeq |\underset{\sim}{x}| - \hat{n}\cdot\underset{\sim}{r}(t')$ for $|\underset{\sim}{x}| \gg |\underset{\sim}{r}|$. Define

$$t_{ret} + \frac{1}{c}R(t_{ret}) = t$$

Recall that $\displaystyle\int dt'\,\delta[f(t')] = \frac{1}{|df/dt|}\Bigg|_{t_{ret}}$

so

$$\underset{\sim}{A} = \frac{\mu_0}{4\pi} \left(\frac{q \underset{\sim}{v}}{R} \frac{1}{\left| 1 + \frac{1}{c} \frac{\partial}{\partial t'} R \right|} \right) \Bigg|_{t' = t_{ret}}$$

$$\frac{\partial}{\partial t'} R = - \hat{n} \cdot \frac{\partial}{\partial t} \underset{\sim}{r}' = - \hat{n} \cdot \underset{\sim}{v}$$

$$\underset{\sim}{A} = \frac{\mu_0}{4\pi} \left(\frac{q \underset{\sim}{v}}{R} \frac{1}{1 - \frac{1}{c} \underset{\sim}{v} \cdot \hat{n}} \right) \Bigg|_{t_{ret}}$$

Similarly for φ,

$$\varphi = \frac{1}{4\pi\epsilon_0} \left[\frac{q}{\left(1 - \frac{1}{c} \hat{n} \cdot \underset{\sim}{v}\right)} \frac{1}{R} \right] \Bigg|_{t_{ret}}$$

$\Rightarrow$ Liénard - Wiechert potentials

$\Rightarrow$ valid for arbitrary $\dot{V}/c$

$\Rightarrow$ Consider $|\dot{\underset{\sim}{v}}|/c \ll 1$ and evaluate $\underset{\sim}{E}$ and $\underset{\sim}{B}$

$$\underset{\sim}{B} = \nabla \times \underset{\sim}{A}$$

$\Rightarrow$ can neglect v/c correction in denominator $\Rightarrow 1 - \frac{1}{c} \hat{n} \cdot \underset{\sim}{v} \simeq 1$

$\Rightarrow$ recall at t_{ret} depends on $|\underset{\sim}{x}|$

$$\underset{\sim}{B} \simeq \frac{\mu_0}{4\pi} \frac{q}{R} \nabla \times \underset{\sim}{v}(t_{ret})$$

$\Rightarrow$ neglect $\nabla \frac{1}{R}$ $\Rightarrow$ radiation field dominates near field $\sim \frac{1}{R^2}$

$$\underset{\sim}{B} \simeq \frac{\mu_0}{4\pi} \frac{q}{R} \nabla(t_{ret}) \times \dot{\underset{\sim}{v}}$$

To calculat $\nabla(t_{ret})$,

$$t_{ret} + \frac{R(t_{ret})}{c} = t$$

$$t_{ret} + \frac{|\underset{\sim}{x}|}{c} - \hat{n} \cdot \underset{\sim}{r}(t_{ret}) \frac{1}{c} = t$$

$$\nabla(t_{ret}) + \frac{\hat{n}}{c} - \hat{n} \cdot \underset{\sim}{v} \frac{1}{c} \nabla(t_{ret}) = 0$$

$$\nabla(t_{ret}) = - \frac{\hat{n} \frac{1}{c}}{1 - \frac{\hat{n} \cdot \underset{\sim}{v}}{c}} \simeq - \frac{\hat{n}}{c}$$

$$\underset{\sim}{B} = - \frac{q}{R} \frac{\mu_0}{4\pi} \frac{1}{c} \hat{n} \times \dot{\underset{\sim}{v}} \Big|_{t_{ret}}$$

Calculate $\underset{\sim}{E}$ from the displacement current,

$$\nabla \times \underset{\sim}{B} = \frac{1}{c^2}\frac{\partial}{\partial t}\underset{\sim}{E}$$

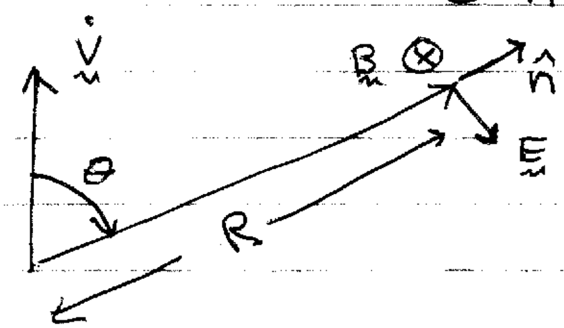
$$\nabla \times \underset{\sim}{B} = -\frac{q}{R}\frac{\mu_0}{4\pi c}\,\nabla(t_{ret}) \times (\hat{n} \times \ddot{\underset{\sim}{v}})$$

$$= \frac{\mu_0}{4\pi}\frac{q}{c^2 R}\,\hat{n} \times (\hat{n} \times \ddot{\underset{\sim}{v}})$$

$$= \frac{\mu_0}{4\pi}\frac{\partial}{\partial t}\frac{q}{c^2 R}\,\hat{n} \times (\hat{n} \times \dot{\underset{\sim}{v}})$$

$$\Longrightarrow \quad \underset{\sim}{E} = \frac{\mu_0}{4\pi}\left[\frac{q}{R}\,\hat{n} \times (\hat{\underset{\sim}{n}} \times \dot{\underset{\sim}{v}})\right]\Bigg|_{t_{ret}}$$

$$= -c\,\hat{n} \times \underset{\sim}{B} \qquad \text{or} \qquad \underset{\sim}{B} = \frac{1}{c}\,\hat{n} \times \underset{\sim}{E}$$



$$E = \frac{\mu_0}{4\pi}\frac{q}{R}\sin\theta\,\dot{v}\,\Big|_{t_{ret}}$$

$$cB = E$$

Poynting flux:

$$\underset{\sim}{S} = \frac{1}{\mu_0}\underset{\sim}{E} \times \underset{\sim}{B} = \frac{1}{\mu_0 c}\,\underset{\sim}{E} \times (\hat{n} \times \underset{\sim}{E})\,\frac{\epsilon_0}{\epsilon_0}$$

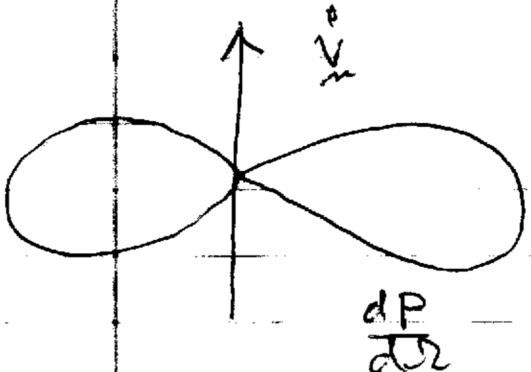
$$= \hat{n}\,\epsilon_0 |\underset{\sim}{E}|^2 c\,\Big|_{t_{ret}}$$

$$\underset{\sim}{S} = \hat{n}\,\epsilon_0 c\left(\frac{\mu_0}{4\pi}\right)^2 \frac{q^2}{R^2}\sin^2\theta\,\dot{v}^2\,\Big|_{t_{ret}}$$

$\Longrightarrow$ radiation maximizes at $\theta = \pi/2$

$$\frac{dP}{d\Omega} = \hat{n} \cdot \underset{\sim}{S} R^2 = \frac{1}{16\pi^2} \epsilon_0^2 \mu_0^2 q^2 \sin^2\theta \, \dot{v}^2 \frac{c}{\epsilon_0}$$

$$\frac{dP}{d\Omega} = \frac{1}{16\pi^2 \epsilon_0 c^3} q^2 \dot{v}^2 \sin^2\theta$$



$$\frac{dP}{d\Omega} \sim \sin^2\theta$$

$\Rightarrow$ peaked $\perp$ to $\dot{\underset{\sim}{v}}$

Can average over solid angle

$$d\Omega = 2\pi \, d(\cos\theta)$$

$$P = \frac{1}{16\pi^2 \epsilon_0 c^3} q^2 \dot{v}^2 \, 2\pi \underbrace{\int_{-1}^{1} d\cos\theta \left(1 - \cos^2\theta\right)}_{2 - \frac{2}{3} = \frac{4}{3}}$$

$$P = \frac{1}{6\pi} \frac{q^2}{\epsilon_0 c^3} \dot{v}^2 \Big|_{t_{ret}}$$

$\Rightarrow$ Larmor formula for radiation from non-relativistic accelerated charge

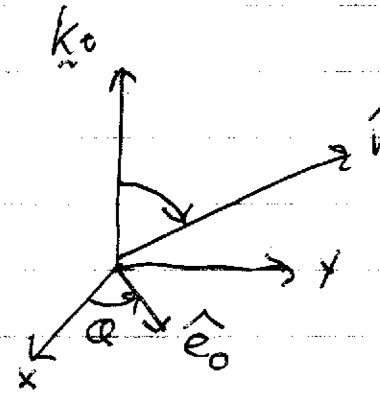
Thomson scattering

An incoming light wave will accelerate electrons. Accelerated electrons will radiate at the frequency of the incident radiation

$\Rightarrow$ scattering of incident radiation

Consider an incident wave $\underset{\sim}{E}_0 = E_0 \hat{e}_0$ the $\underset{\sim}{k}_0$ along z direction

$$\dot{V}_0 = -\frac{e}{m_e} E_0 \hat{e}_0 e^{ik_0 z} e^{-i\omega t}$$



$$\langle \dot{V}^2 \rangle_t = \frac{1}{2} \dot{V}_0 \cdot \dot{V}_0^*$$

$$\frac{dP}{d\Omega} = \frac{e^2}{16\pi^2 \epsilon_0 c^3} \frac{1}{2} \frac{e^2}{m^2} |\underset{\sim}{E}_0|^2 \otimes |\hat{n} \times \hat{e}_0|^2$$

Choose $\hat{n} = (0, \sin\theta, \cos\theta)$

$$\hat{e}_0 = (\cos\varphi, \sin\varphi, 0)$$

$$|\hat{n} \times \hat{e}_0|^2 = \cos^2\theta + \sin^2\theta \cos^2\varphi$$

For unpolarized light, $\hat{e}_0$ can have any angle

$\Rightarrow$ average over φ

$$\langle |\hat{n} \times \hat{e}_0|^2 \rangle_\varphi = \cos^2\theta + \sin^2\theta \, \frac{1}{2}$$

$$= \frac{1}{2}\left(1 + \cos^2\theta\right)$$

Define a scattering cross section

$$\frac{d\sigma}{d\Omega} = \frac{\text{energy radiated / time - solid angle}}{\text{incident energy / area - time}}$$

$$\sim \frac{\text{area}}{\text{solid angle}}$$

$$\frac{d\sigma}{d\Omega} = \frac{e^2}{16\pi^2 \epsilon_0 c^3} \frac{\frac{1}{2} \frac{e^2}{m^2} |\underset{\sim}{E}_0|^2 \left(1+\cos^2\theta\right)}{\frac{1}{2} \epsilon_0 |\underset{\sim}{E}_0|^2 c}$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{4\pi \epsilon_0 c^2 m}\right)^2 \frac{1}{2}\left(1+\cos^2\theta\right)$$

$\Rightarrow$ Thomsen formula

Can average over solid angle to find total cross section.

$$\sigma_T = \left(\frac{e^2}{4\pi\epsilon_0 c^2}\right)^2 \frac{2\pi}{2}\left(1+\frac{2}{3}\right)$$

$$= \frac{8\pi}{3}\left(\frac{e^2}{4\pi\epsilon_0 m c^2}\right)^2$$

= Thomson cross section

$= .665 \times 10^{-24}\ cm^2$

$\frac{e^2}{4\pi\epsilon_0 m c^2}$ = classical electron radius

$\Rightarrow$ radius at which electron potential energy is equal to rest mass energy.