Electromagnetic Wave Propagation

Consider waves propagating in a homogeneous, non-conducting medium. Maxwell's Eqns give

$$\nabla\cdot\underline{E}=0 \qquad \nabla\times\underline{E}+\cancel{\frac{1}{c}}\frac{\partial}{\partial t}\underline{B}=0 \qquad (1)$$

$$\nabla\cdot\underline{B}=0 \qquad \nabla\times\underline{B}-\frac{\mu\epsilon}{\cancel{c}}\frac{\partial}{\partial t}\underline{E}=0 \qquad (2)$$

Take the curl of (1)

$$\nabla\times(\nabla\times\underline{E}) = \nabla\,\cancel{\nabla\cdot\underline{E}}^{\,0} - \nabla^2\underline{E} = -\cancel{\frac{1}{c}}\frac{\partial}{\partial t}\frac{\mu\epsilon}{\cancel{c}}\frac{\partial}{\partial t}\underline{E}$$

$$\left(\nabla^2 - \frac{\mu\epsilon}{\cancel{c^2}}\frac{\partial^2}{\partial t^2}\right)\underline{E}=0 \implies \text{wave eqn.}$$

$\implies$ same for $\underline{B}$

$\implies$ exponential solutions

$$\underline{E} = \mathrm{Re}\left(\underline{E}_0\, e^{i\underline{k}\cdot\underline{x}-i\omega t}\right)$$

$$= \frac{1}{2}\left[\underline{E}_0 e^{i\underline{k}\cdot\underline{x}-i\omega t} + \underline{E}_0^* e^{-i\underline{k}\cdot\underline{x}+i\omega t}\right]$$

$$\left(k^2 - \frac{\mu\epsilon}{\cancel{c^2}}\omega^2\right)\underline{E}_0 = 0 \qquad c = \frac{1}{\sqrt{\mu_0\epsilon_0}}$$

$$\boxed{\omega = \pm k v_p} \qquad v_p = \frac{\cancel{c}}{\sqrt{\mu\epsilon}} = \frac{c}{n}$$

dispersion relation

n = index of refraction

$n = \sqrt{\mu\epsilon/\mu_0\epsilon_0}$

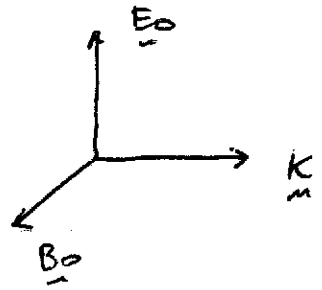
To find vector direction of $\underset{\sim}{E}$ and $\underset{\sim}{B}$

$$\cancel{i}\, \underset{\sim}{k} \times \underset{\sim}{E}_0 - \cancel{i}\,\frac{\omega}{c} \underset{\sim}{B}_0 = 0 \quad , \quad \underset{\sim}{k}\cdot \underset{\sim}{E}_0 = 0$$

$$\underset{\sim}{k}\cdot \underset{\sim}{B}_0 = 0 \qquad \omega = \frac{kc}{n}$$

$$\underset{\sim}{B}_0 = \frac{c}{\omega}\, \underset{\sim}{k} \times \underset{\sim}{E}_0 \qquad B_0 = \frac{n E_0}{c}$$

for $\omega > 0$



$$\underset{\sim}{S} = \cancel{\frac{c}{4\pi}}\, \underset{\sim}{E} \times \underset{\sim}{H}$$

$\Rightarrow$ $\underset{\sim}{S}$ is in the direction of $\underset{\sim}{k}$ and is $\perp$ to $\underset{\sim}{E}$, $\underset{\sim}{B}$

Let $\delta = \underset{\sim}{k}\cdot\underset{\sim}{x} - \omega t$

$\Rightarrow$ flow of energy is along $\underset{\sim}{k}$

$\Rightarrow$ transverse wave

$$\underset{\sim}{S} = \cancel{\frac{c}{4\pi}} \left(\frac{\underset{\sim}{E}_0 e^{i\delta} + \underset{\sim}{E}_0^* e^{-i\delta}}{2} \right) \times \left(\frac{\underset{\sim}{H}_0 e^{i\delta} + \underset{\sim}{H}_0^* e^{-i\delta}}{2} \right)$$

$$= \frac{1}{4} \cancel{\frac{c}{4\pi}} \Big[\underset{\sim}{E}_0 \times \underset{\sim}{H}_0 e^{2i\delta} + \underset{\sim}{E}_0^* \times \underset{\sim}{H}_0^* e^{-2i\delta} + \underset{\sim}{E}_0 \times \underset{\sim}{H}_0^* + \underset{\sim}{E}_0^* \times \underset{\sim}{H}_0 \Big)$$

$\Rightarrow$ time or space average is important quantity

$$\boxed{\bar{\underset{\sim}{S}} = \cancel{\frac{c}{4\pi}}\ \frac{1}{2} \mathrm{Re} \left(\underbrace{E_0 \times H_0^*}_{\frac{1}{\mu}\frac{c}{\omega} \underset{\sim}{E}_0 \times (\underset{\sim}{k} \times E_0^*)} \right)}$$

General rule for evaluating time-averaged products

$$\overline{\underset{\sim}{C}\,\underset{\sim}{D}} = \frac{1}{2}\,\mathrm{Re}(\underset{\sim}{C}_0 D_0^*)$$

$$\overline{\underset{\sim}{S}} = \frac{1}{2}\,\underset{\sim}{k}\,|\underset{\sim}{E}_0|^2\,\frac{1}{\mu\omega} = \frac{1}{2}\,\underset{\sim}{k}\,|\underset{\sim}{E}_0|^2\,\frac{n}{\mu k c}$$

$$= \frac{1}{2}\,\frac{\underset{\sim}{k}}{k}\,\epsilon|\underset{\sim}{E}_0|^2\,\frac{n}{\underbrace{\mu\epsilon c}_{c/n = V_p}}$$

$$\overline{\underset{\sim}{S}} = \frac{1}{2}\,\frac{\underset{\sim}{k}}{k}\,\epsilon|\underset{\sim}{E}_0|^2\,V_p$$

Want to express this in terms of the wave energy

$$u = \frac{1}{2}\left(\epsilon|\underset{\sim}{E}|^2 + \frac{1}{\mu}|\underset{\sim}{B}|^2\right)$$

$$\bar{u} = \frac{1}{4}\left(\epsilon|\underset{\sim}{E}_0|^2 + \frac{1}{\mu}|\underset{\sim}{B}_0|^2\right)$$

$$\frac{1}{\mu}|\underset{\sim}{B}_0|^2 = \frac{1}{\mu}\,\frac{k^2}{\omega^2}|\underset{\sim}{E}_0|^2 = \epsilon|\underset{\sim}{E}_0|^2$$

$\Rightarrow$ magnetic and electric wave energies are equal

$$\bar{u} = \frac{1}{2}\,\epsilon|\underset{\sim}{E}_0|^2$$

$$\overline{\underset{\sim}{S}} = \bar{u}\,\frac{\underset{\sim}{k}}{k}\,V_p$$

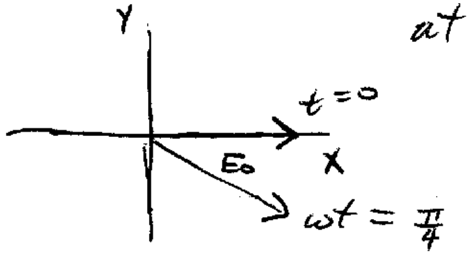
polarization $\Rightarrow$ defined by direction of $\underset{\sim}{E}$

plane polarized $\Rightarrow$ $\underset{\sim}{E}$ in single direction

circularly polarized $\Rightarrow$

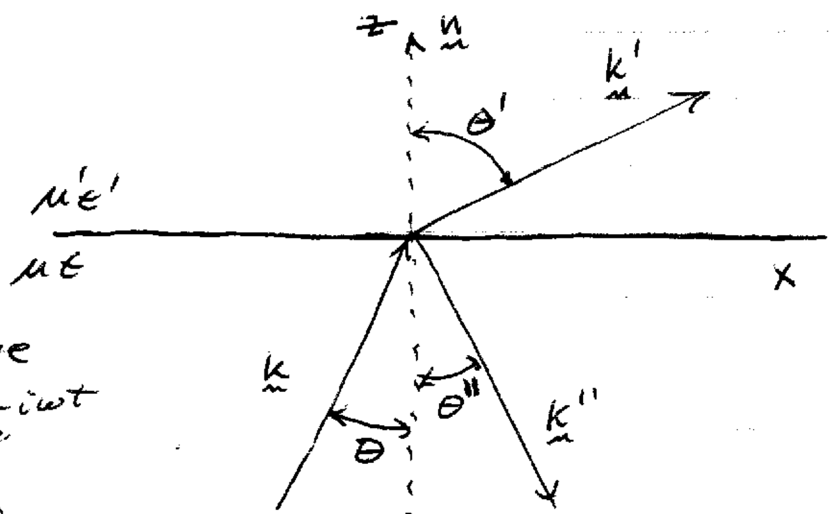
$$\underset{\sim}{E} = E_0 \left[\hat{\imath} \cos(kz - \omega t) + \hat{\jmath} \sin(kz - \omega t) \right]$$

at $z = 0$



$\Rightarrow$ ~~right~~ left hand circularly polarized.

Reflection and refraction of EM waves from a plane interface between dielectrics.



Incident wave

$\underset{\sim}{E}_0 e^{i \underset{\sim}{k} \cdot \underset{\sim}{x}} e^{-i\omega t}$

$B_0 = \frac{n E_0}{c}$

Wave creates oscillating polarization charges and magnetization currents which cause reflected waves and alters direction of transmitted wave

~~Prefactory~~

What are the frequencies of the transmitted and reflected waves?

$$\Rightarrow \omega \quad \text{(homogeneity in time)}$$

Each wave must satisfy local dispersion relation

$$\omega = \omega'' \Rightarrow \boxed{k = k''} = \frac{\omega n}{c}$$

$$\omega = \cancel{b}\,\frac{kc}{n} = \omega' = \frac{k'c}{n'}$$

$$\boxed{k' = \frac{n'}{n} k}$$

Boundary Conditions

$\mathbf{B} \cdot \mathbf{n}$ continuous ①

$\mathbf{D} \cdot \mathbf{n}$ continuous ②

$\mathbf{H} \times \mathbf{n}$ continuous ③

$\mathbf{E} \times \mathbf{n}$ continuous ④

B.C. must be matched along entire interface

$\Rightarrow$ phases must match at $z = 0$

$$\sim e^{ik_x x} e^{-i\omega t}$$

$$\Rightarrow k_x = k_x' = k_x''$$

$\Rightarrow k_y = 0 \Rightarrow \mathbf{k}$ vectors form a plane

$$k_x = k_x'' \quad \text{with} \quad k = k'' \quad \text{implies} \quad \boxed{\theta = \theta''}$$

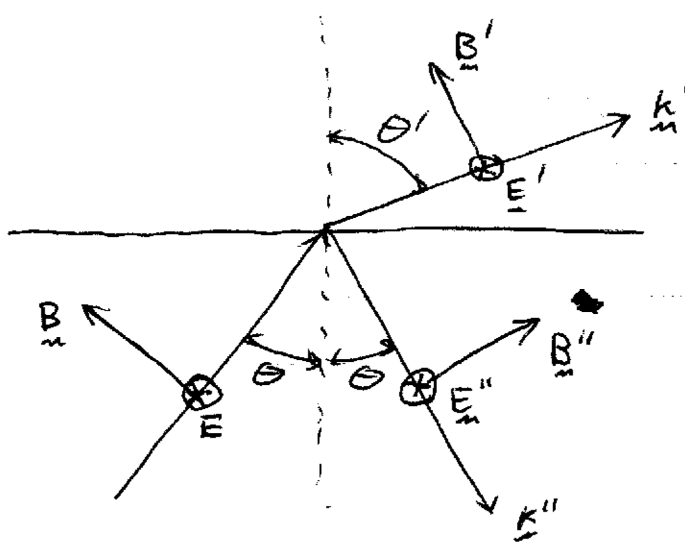
$\Longrightarrow$ angle of reflection equals angle of incidence

$$k_x = k_x' \Rightarrow \quad k \sin\theta = k' \sin\theta'$$

$$k \sin\theta = \frac{n'}{n} k \sin\theta'$$

$$\boxed{n \sin\theta = n' \sin\theta'} \quad \text{Snell's Law}$$

<u>Case I</u> Polarization $\perp$ to Plane



② Gives nothing

④ gives

$$E_0 + E_0'' = E_0'$$

③ gives ($\underline{H} \times \hat{n}$ continuous) $\quad \cancel{\frac{B_{x0}}{\mu}}$

$$B = \frac{nE}{c}$$

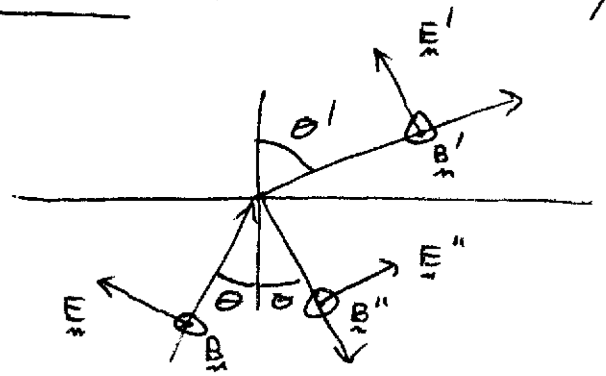
$$-\frac{B_0}{\mu}\cos\theta + \frac{B_0''}{\mu}\cos\theta = -\frac{B_0'}{\mu'}\cos\theta'$$

$$n\frac{\cos\theta}{\mu}\left(E_0'' - E_0\right) = -\frac{\cos\theta'}{\mu'}n' E_0'$$

$$\frac{E_0'}{E_0} = \frac{2n\cos\theta}{n\cos\theta + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2\theta}}$$

$$\frac{E_0''}{E_0} = \frac{n\cos\theta - \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2\theta}}{n\cos\theta + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2\theta}}$$

Case II Polarization in plane



$$\frac{E_0'}{E_0} = \frac{2nn'\cos\theta}{\frac{\mu}{\mu'}n'^2\cos\theta + n\sqrt{n'^2 - n^2\sin^2\theta}}$$

$$\frac{E_0''}{E_0} = \frac{\frac{\mu}{\mu'}n'^2\cos\theta - n\sqrt{n'^2 - n^2\sin^2\theta}}{\frac{\mu}{\mu'}n'^2\cos\theta + n\sqrt{n'^2 - n^2\sin^2\theta}}$$

Take $\theta = 0$ and $\mu = \mu'$

$$\frac{E_0'}{E_0} = \frac{2nn'}{n'^2 + nn'} = \frac{2n}{n'+n}$$

$$\frac{E_0''}{E_0} = \frac{n'^{\cancel{2}} - n\cancel{n'}}{n'^{\cancel{2}} + n\cancel{n'}} = \frac{n'-n}{n'+n}$$

phase of reflected wave
depends on relative size of n', n

Brewsters Angle

Consider $\underset{\sim}{E}$ in plane of incidence

Can have $E_0'' = 0$ (take $\mu = \mu' = \mu_0$)

$$n'^2 \cos i = n\sqrt{n'^2 - n^2 \sin^2 i}$$

$$\frac{n'^2}{n^2}\cos i = \sqrt{\frac{n'^2}{n^2} - \sin^2 i} \qquad f = \frac{n'^2}{n^2}$$

$$f^2 \cos^2 i = f - \sin^2 i \qquad s^2 + c^2 = 1$$

$$t^2 + 1 = \frac{1}{c^2}$$

$$f^2 = \frac{f}{c^2} - t^2$$

$$f^2 = f(1+t^2) - t^2$$

$$f^2 - f(1+t^2) + t^2 = 0$$

$$f = \frac{1+t^2 \pm \sqrt{(1+t^2)^2 - 4t^2}}{2} = \frac{1+t^2 \pm (1-t^2)}{2}$$

$$f = 1,\ t^2 \Rightarrow \boxed{n = n'} \qquad \boxed{\frac{n'}{n} = \tan i_B}$$

No reflection at $i = i_B$.

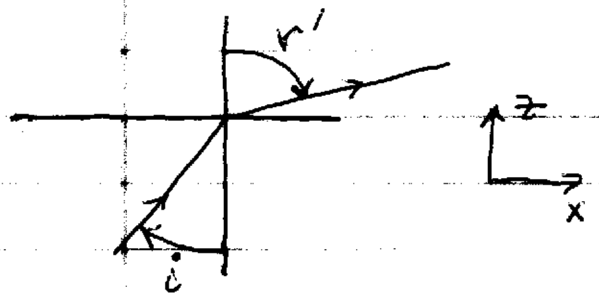
At this angle light reflected from a dielectric is polarized perpendicular to the plane of incidence

$\Rightarrow$ how do polarized sunglasses work?

<u>Total internal reflection</u> $\quad n' < n$

For $\sin(i) > \frac{n'}{n} \Rightarrow \underbrace{\sin(r') = \sin(i)\frac{n}{n'}}_{>1}$

$\Rightarrow r'$ is complex



$$\cos(r') = \left[1 - \sin^2(r')\right]^{1/2}$$
$$= \left[1 - \frac{\sin^2(i)\, n^2}{n'^2}\right]^{1/2}$$
$$= \underbrace{i}_{\sqrt{-1}}\left[\frac{n^2}{n'^2}\sin^2(i) - 1\right]^{1/2}$$

$$\underset{\sim}{E}' = \underset{\sim}{E}_0' \, e^{i\underset{\sim}{k}'\cdot\underset{\sim}{x} - i\omega t} \quad \Rightarrow \underset{\sim}{k}' \text{ is complex}$$

$$k_x' = k'\sin(r') = k'\frac{n}{n'}\sin(i)$$
$$k_z' = k'\cos(r') = i\,k'\left[\frac{n^2}{n'^2}\sin^2(i) - 1\right]^{1/2}$$

$$\underset{\sim}{E}' = \underset{\sim}{E}_0' \, e^{i k_x' x} \, e^{-k' z\left[\frac{n^2}{n'^2}\sin^2(i) - 1\right]^{1/2}}$$

$\Rightarrow$ the transmitted wave falls off exponentially as z increases.

Is energy dissipated in the n' medium or is there no energy transmission?

$\Rightarrow$ evaluate the Poynting flux

$$\underset{\sim}{S}\cdot\hat{n} = \frac{1}{\mu'}\,\underset{\sim}{E}'\times\underset{\sim}{B}'\cdot\hat{n}$$

Need to do a time average so write $\underset{\sim}{E}'$ and $\underset{\sim}{B}'$ in terms of amplitude and phase $\delta = \underset{\sim}{k}'\cdot\underset{\sim}{x} - \omega t$

$$\underset{\sim}{E}' = \left[\underset{\sim}{E}_0 e^{i\delta} + \underset{\sim}{E}_0^* e^{-i\delta^*}\right]\frac{1}{2} \Rightarrow \text{ note that } \delta \neq \delta^*$$

$$\underset{\sim}{S}\cdot\hat{n} = \frac{\hat{n}}{4}\cdot\left[\left(\underset{\sim}{E}_0 e^{i\delta} + \underset{\sim}{E}_0^* e^{-i\delta^*}\right)\times\left(B_0 e^{i\delta} + B_0^* e^{-i\delta^*}\right)\right]$$

$\Rightarrow$ average over time $\Rightarrow$ only $\delta - \delta^*$ terms survive

$$\overline{\underset{\sim}{S}}\cdot\hat{n} = \frac{\hat{n}}{4}\cdot\left[E_0\times B_0^* e^{i(\delta-\delta^*)} + \underset{\sim}{E}_0^*\times\underset{\sim}{B}_0 e^{i(\delta-\delta^*)}\right]$$

$$\delta - \delta^* = 2ik'[\;\;]^{1/2} \equiv 2|k_z|i$$

$$e^{i(\delta-\delta^*)} = e^{-2k'[\;\;]^{1/2}z} = e^{-2|k_z'|z}$$

$$\overline{\underset{\sim}{S}}\cdot\hat{n} = \frac{\hat{n}}{4}e^{-2|k_z'|z}\underbrace{\left(E_0\times B_0^* + E_0^* + \underset{\sim}{B}_0\right)}_{2\,\mathrm{Re}(E_0\times B_0^*)}$$

where $\quad \underset{\sim}{B}_0^* = \frac{1}{\omega}\,\underset{\sim}{k}'^*\times\underset{\sim}{E}_0^*$

$$\underset{\sim}{\bar{S}} \cdot \hat{n} = \frac{\hat{n}}{2} e^{-2|k_z'|z} \cdot \mathrm{Re}\Big[\underbrace{\underset{\sim}{E_0} \times \left(\tfrac{1}{\omega} \underset{\sim}{k}'^* \times \underset{\sim}{E_0}^*\right)}_{\frac{1}{\omega} \underset{\sim}{k}'^* \underbrace{\underset{\sim}{E_0} \cdot \underset{\sim}{E_0}^*}_{\text{real}}} \Big]$$

$$= \frac{1}{2} \underset{\sim}{E_0} \cdot \underset{\sim}{E_0}^* \frac{e^{-2|k_z'|z}}{\omega} \mathrm{Re}\big(\underbrace{\underbrace{\hat{n} \cdot k'^*}_{k_z'^*}}_{\Rightarrow k_z' \text{ is imaginary}} \big)$$

$$\Rightarrow \underset{\sim}{\hat{S}} \cdot \hat{n} = 0$$

Can evaluate the amplitude E_0'' of the reflected wave,

$$\frac{E_0''}{E_0} = \frac{n\cos\theta_i - \frac{\mu}{\mu'} i \left(n^2 \sin^2\theta_i - n'^2\right)^{\frac{1}{2}}}{n\cos\theta_i + \frac{\mu}{\mu'} i \left(n^2 \sin^2\theta_i - n'^2\right)^{\frac{1}{2}}}$$

$$\Rightarrow |E_0''| = |E_0|$$

$\Rightarrow$ All energy reflected from surface

$\Rightarrow$ $\underset{\sim}{E_0}''$ has a phase shift. This can be used to convert light to circular polarization

Wave propagation in a conductor

In a good conductor there will be free currents that flow in response to the electric field of the wave. Maxwell's eqns are given by

$$\nabla \times H = \underset{\sim}{J} + \frac{\partial}{\partial t} \underset{\sim}{D}$$

$$\nabla \times \underset{\sim}{E} + \frac{\partial}{\partial t} \underset{\sim}{B} = 0$$

Since we a interested in transverse waves $\nabla \cdot \underset{\sim}{D} = 0$ so there is no free charge. We calculate $\underset{\sim}{J}$ from the electron momentum eqn,

$$m_e \frac{d}{dt} \underset{\sim}{V_e} = -e \underset{\sim}{E} - m_e \nu_{ei} (\underset{\sim}{V_e} - \underset{\sim}{V_i})$$

with ν_{ei} = electron/ion collision rate. The ion velocity can be neglected ~~since their velocity is small because of their much greater mass~~ since they form a stationary lattice.

$$m_e \frac{d}{dt} \underset{\sim}{V_e} = -e \underset{\sim}{E} - m_e \nu_{ei} \underset{\sim}{V_e}$$

Typically $\frac{\partial}{\partial t} \ll \nu_{ei}$ so electron inertia can also be neglected

$\Rightarrow$ for copper $\nu_{ei} \sim 10^{13}/s$ so this requires $\omega < 10^{13}/s$

$$\Rightarrow \quad \underset{\sim}{V}_e = -\frac{e\underset{\sim}{E}}{m_e \nu_{ei}}$$

and

$$\underset{\sim}{J} = -ne\underset{\sim}{V}_e = \frac{ne^2}{m_e \nu_{ei}} \cdot \underset{\sim}{E} \equiv \sigma \underset{\sim}{E}$$

with $n =$ # of free electrons / unit vol.

$$\sigma = \text{conductivity} \sim 5.9 \times 10^7 \frac{1}{\Omega m}$$

for Cu.

units: $\Omega = \text{Ohm} = \frac{\text{Volt}}{\text{amp}} = \frac{Nm}{C^2} s$

$$\nabla \times \underset{\sim}{B} = \mu\sigma \underset{\sim}{E} + \mu\epsilon \frac{\partial}{\partial t} \underset{\sim}{E}$$

$$\nabla \times \underset{\sim}{E} + \frac{\partial}{\partial t} \underset{\sim}{B} = 0$$

As before take $\underset{\sim}{E}, \underset{\sim}{B} \sim e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}} e^{-i\omega t}$

$$i\underset{\sim}{k} \times \underset{\sim}{B}_0 = \mu\sigma \underset{\sim}{E}_0 - \mu\epsilon i\omega \underset{\sim}{E}_0$$

$$\cancel{i}\underset{\sim}{k} \times \underset{\sim}{E}_0 - \cancel{i}\omega \underset{\sim}{B}_0 = 0$$

$$\frac{1}{\omega} \underbrace{\underset{\sim}{k} \times (\underset{\sim}{k} \times \underset{\sim}{E}_0)}_{-k^2 \underset{\sim}{E}_0} = (-i\mu\sigma - \mu\epsilon\omega) \underset{\sim}{E}_0$$

$$k^2 = \omega^2 \mu \epsilon \left(1 + i \frac{\sigma}{\omega \epsilon}\right)$$

$\Rightarrow$ k is now complex
$\Rightarrow$ dissipation of wave energy from collisions
$\Rightarrow$ $\underset{\sim}{J} \cdot \underset{\sim}{E} = \sigma E^2 > 0$

compare 1, $\frac{\sigma}{\omega \epsilon}$

$$\frac{\sigma}{\omega \epsilon} \sim \frac{6 \times 10^7}{\Omega m} \frac{1 \, Nm^2}{9 \times 10^{-12} C^2 \omega}$$

$$\sim \frac{6 \times 10^7}{9 \times 10^{-12}} \frac{1}{\omega} \frac{Nm^2 \, C^2}{C^2 \, \frac{Nm}{s} m} \sim 10^{19} \frac{1}{\omega s}$$

Since already assumed $\omega < 10^{13}/s$,

$$\frac{\sigma}{\omega \epsilon} \gg 1 \quad \Rightarrow \sigma E \gg \epsilon \frac{\partial}{\partial t} E$$

$\Rightarrow$ $\underset{\sim}{J}$ dominates displacement current

$$\Rightarrow k^2 = i \omega \mu \sigma$$

$$k = (1+i) \sqrt{\frac{\omega \mu \sigma}{2}} \equiv (1+i) \frac{1}{\delta}$$

$$\delta = \text{skin depth} = \sqrt{\frac{2}{\omega \mu \sigma}}$$

$$\underset{\sim}{E} = E_0\, e^{i\left(\frac{z}{\delta} - \omega t\right)}\, e^{-\frac{z}{\delta}}$$

$\Rightarrow$ skin depth is the characteristic dissipation scale length in a conductor

For copper, $\delta \sim 10^{-3}$ cm for $\omega = 100$ MHz

$\Rightarrow$ waves don't penetrate very far into a good conductor

Why are the waves evanescent?
Waves are doing work on electrons at a rate

$$\underset{\sim}{J} \cdot \underset{\sim}{E} = \sigma |\underset{\sim}{E}|^2 \Rightarrow \text{Joule heating}$$

$\Rightarrow$ energy absorbed by the medium

Compare δ with the free space wavelength

$$\Rightarrow k_0 = \frac{\omega}{c}, \quad \mu, \epsilon \sim \mu_0, \epsilon_0$$

$$k_0 \delta = \frac{\omega}{c} \sqrt{\frac{2}{\omega \mu \sigma}} \sim \sqrt{\mu_0 \epsilon_0} \sqrt{\frac{\omega}{\mu_0 \sigma}}$$

$$\sim \sqrt{\frac{\omega}{\sigma} \epsilon_0} \ll 1$$

$\Rightarrow \delta$ shorter than free space wavelength

Wave propagation in a collisionless plasma

A plasma is a gas of electrons and ions.

$\Rightarrow$ must calculate $\underset{\sim}{J}$ since electrons are free to move

$\Rightarrow$ electron current dominates that of ions because of smaller mass.

$$m_e \frac{d}{dt} \underset{\sim}{V_e} = -e \underset{\sim}{E}$$

$$-i m_e \omega \underset{\sim}{V_{e0}} = -e \underset{\sim}{E_0}$$

$$\Rightarrow \underset{\sim}{V_{e0}} = \frac{e \underset{\sim}{E_0}}{i \omega m_e}$$

$$\underset{\sim}{J_0} = -ne \underset{\sim}{V_{e0}} = -\frac{ne^2}{m_e i\omega} \underset{\sim}{E_0}$$

$\Rightarrow$ again, no charge ρ since $\underset{\sim}{k} \cdot \underset{\sim}{E_0} = 0$

$\Rightarrow$ transverse waves

$$i \underset{\sim}{k} \times \underset{\sim}{B_0} = \mu_0 \underset{\sim}{J_0} - \frac{1}{c^2} i \omega \underset{\sim}{E_0}$$

$$\underset{\sim}{k} \times \underset{\sim}{E_0} - \omega \underset{\sim}{B_0} = 0$$

$$\underbrace{\underset{\sim}{k} \times (\underset{\sim}{k} \times \underset{\sim}{E_0})}_{-k^2 \underset{\sim}{E_0}} \frac{1}{\omega} = -i\mu_0 \left(-\frac{ne^2}{m_e i\omega}\right) \underset{\sim}{E_0} - \frac{\omega}{c^2} \underset{\sim}{E_0}$$

$$\omega^2 = \omega_{pe}^2 + k^2 c^2$$ plasma dispersion relation

$$\omega_{pe} = \left(\frac{ne^2}{m_e \epsilon_0}\right)^{1/2} = \text{plasma frequency}$$

$$= 5.6 \times 10^4 \sqrt{n} \,/\text{sec}$$

with n in units of cm^{-3}.

In the solar wind at 1AU, $n \sim 1/cm^3$

$\Rightarrow \omega_{pe} \sim 6 \times 10^4/s$

Note that in a collisionless plasma $\underset{\sim}{k}$ is real for $\omega > \omega_{pe}$. Why?

$$\overline{\underset{\sim}{J_1} \cdot \underset{\sim}{E}} = \frac{1}{2} \operatorname{Re}(\underset{\sim}{J_0} \cdot \underset{\sim}{E_0}^*)$$

$$\sim \operatorname{Re}(i \underset{\sim}{E_0} \cdot \underset{\sim}{E_0}^*) = 0$$

$\Rightarrow$ no electron heating

$\Rightarrow$ no dissipation

For $\omega < \omega_{pe}$,

$$kc = \sqrt{\omega^2 - \omega_{pe}^2} = i\sqrt{\omega_{pe}^2 - \omega^2}$$

$\Rightarrow$ wave reflection but no dissipation

$\Rightarrow$ like total internal reflection

Group versus phase velocity

For a plasma wave

$$\omega^2 = \omega_{pe}^2 + k^2c^2$$

Consider the phase of a wave,

$$E \sim e^{ikx - i\omega t} = e^{i\delta}$$

Velocity of phase of the wave, V_p,

$$\frac{d}{dt}\delta = k\dot{x} - \omega = 0$$

$$V_p = \dot{x} = \frac{\omega}{k}$$

For a plasma wave

$$\frac{V_p}{c} = \frac{\sqrt{\omega_{pe}^2 + k^2c^2}}{kc} = \sqrt{1 + \frac{\omega_{pe}^2}{k^2c^2}} > 1$$

$\Rightarrow V_p > c$

$\Rightarrow$ How is this possible?

$\Rightarrow$ no information propagation at V_p

$\Rightarrow$ must modulate the signal.

$\Rightarrow$ modulated signal propagates at the group velocity.

Group velocity

Consider 1-D wave propagation

$$E_x,\ B_y,\ \frac{\partial}{\partial z} \neq 0$$

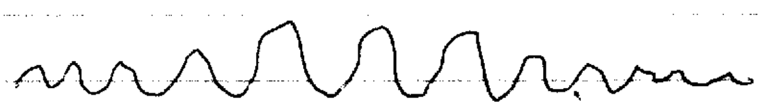
$$\frac{\partial}{\partial t} \underset{\sim}{B} + \nabla \times \underset{\sim}{E} = 0$$

$$(1) \quad \frac{\partial}{\partial t} B_y + \frac{\partial}{\partial z} E_x = 0$$

$$\nabla \times \underset{\sim}{B} = \mu_0 \underset{\sim}{J} + \mu_0 \epsilon_0 \frac{\partial}{\partial t} \underset{\sim}{E}$$

$$(2) \quad -\frac{\partial}{\partial z} B_y = \mu_0 J_x + \frac{1}{c^2} \frac{\partial}{\partial t} E_x$$

Consider a nearly monochromatic wave



$\Rightarrow$ modulation transfers information

$$E_x(z,t) = E_0(z,t) \cos(k_0 z - \omega_0 t)$$

with $E_0(z,t)$ slowly varying in space and time.

$$\frac{\partial}{\partial z} E_0 \ll k_0 E_0$$

$$\frac{\partial}{\partial t} E_0 \ll \omega_0 E_0$$

Take $\frac{\partial}{\partial t}$ of eqn. (2)

$$-\frac{\partial}{\partial z}\frac{\partial}{\partial t} B_y = -\frac{\partial}{\partial z}\left(-\frac{\partial}{\partial z} E_x\right)$$

$$= \mu_0 \underbrace{\frac{\partial}{\partial t} J_x}_{-ne \underbrace{\frac{\partial}{\partial t} V_{ex}}_{-\frac{e}{m_e} E_x}} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2} E_x$$

$$\frac{\partial^2}{\partial t^2} E_x - c^2 \frac{\partial^2}{\partial z^2} E_x + \omega_{pe}^2 E_x = 0$$

$$\ddot{E}_x = -\omega_0^2 E_0(z,t) \cos(\ ) + 2\omega_0 \dot{E}_0 \sin(\ ) + \cancel{\ddot{E}_0 \cos(\ )}^{\text{small}}$$

$$\frac{\partial^2}{\partial z^2} E_x = -k_0^2 E_0 \cos(\ ) - 2k_0\left(\frac{\partial}{\partial z} E_0\right)\sin(\ ) + \cancel{\frac{\partial^2 E_0}{\partial z^2}\cos(\ )}^{\text{small}}$$

$\Rightarrow$ keep only first order in derivatives acting on E_0.

$$\left(\cancel{-\omega_0^2 + k_0^2 c^2 + \omega_{pe}^2}^{\,0}\right) E_0 \cos(\ )$$

$$+ 2\left(\omega_0 \dot{E}_0 + k_0 c^2 \frac{\partial}{\partial z} E_0\right)\sin(\ ) = 0$$

$$\frac{\partial}{\partial t} E_0 + \frac{K_0 c^2}{\omega_0} \frac{\partial}{\partial z} E_0 = 0$$

$$\omega_0^2 = \omega_{pe}^2 + k_0^2 c^2$$

$$2\,\omega_0 \underbrace{\frac{d\omega_0}{dk_0}}_{V_g} = 2 k_0 c^2$$

$$\frac{d\omega_0}{dk_0} = V_g = \frac{k_0 c^2}{\omega_0} = \text{group velocity}$$

$$\frac{\partial}{\partial t} E_0 + V_g \frac{\partial}{\partial z} E_0 = 0$$

$$\Rightarrow \quad E_0(z,t) = E_0(z - V_g t, 0)$$

$\Rightarrow$ wave modulation propagates at V_g

For plasma waves

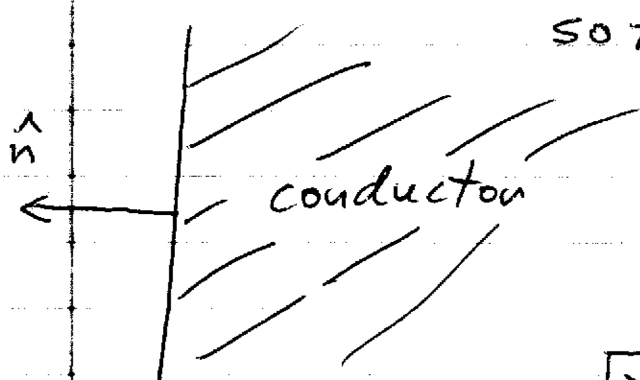
$$\frac{V_g}{c} = \frac{k_0 c}{\omega_0} = \frac{c}{V_p} < 1$$

$\Rightarrow$ information propagates at a velocity less than c.

Fields at the surface of a conductor

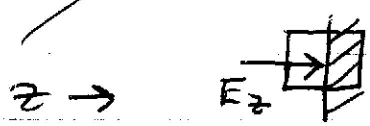
First consider a <u>perfect</u> <u>conductor</u> with $\sigma \to \infty$.

$\Rightarrow$ $\underset{\sim}{E} = 0$ inside the conductor so that $\underset{\sim}{J} \neq \infty$.



From $\nabla \cdot \underset{\sim}{D} = \rho$

$\hat{n} \cdot \underset{\sim}{D} = \Sigma =$ surface charge density



$\Rightarrow$ at the surface $E_z \neq 0$.

From Faraday's law

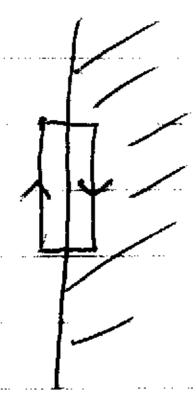
$$\frac{\partial}{\partial t}\underset{\sim}{B} = -\nabla \times \underset{\sim}{E} \quad \Rightarrow \quad \underset{\sim}{B} = 0$$ inside conductor for a time varying wave.

$\Rightarrow$ $\hat{n} \cdot \underset{\sim}{B} = 0$

$\Rightarrow$ $B_z = 0$ at the surface

From

$$\nabla \times \underset{\sim}{H} = \underset{\sim}{J} + \epsilon_0 \frac{\partial}{\partial t}\underset{\sim}{E}$$



From integral around loop and $\underset{\sim}{E}$ zero or finite inside loop

$$\hat{n} \times \underset{\sim}{H} = \underset{\sim}{K} = \text{surface current}$$

$$\Rightarrow \text{tangential } \underset{\sim}{H} \neq 0$$

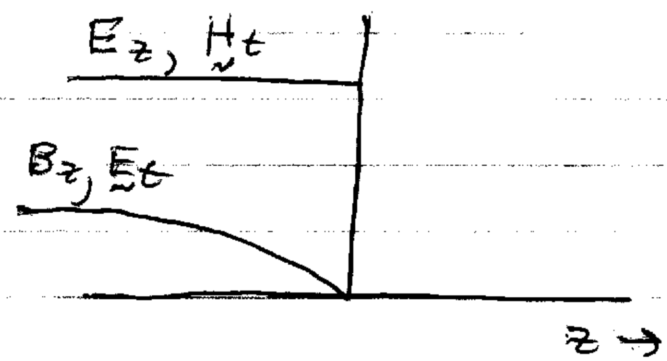
$$\Rightarrow \underset{\sim}{H}_t \neq 0$$

From $\nabla \times \underset{\sim}{E} + \frac{\partial}{\partial t}\underset{\sim}{B} = 0$ integrated around loop and $\underset{\sim}{B}$ finite (outside) and zero inside

$$\hat{n} \times \underset{\sim}{E} = 0$$

$$\Rightarrow \underset{\sim}{E}_t = 0$$

$\Rightarrow$ field profiles at the surface of an ideal conductor



Now consider a non-ideal conductor with finite σ and small but non zero skin depth δ.

$\Rightarrow$ EM waves decay over a distance δ that is small compared with $k_0 = c/\omega$.

$\Rightarrow \underset{\sim}{E} \to 0$ once are at a distance greater than δ in the conductor

$\Rightarrow$ very close to the surface $\underset{\sim}{E} \neq 0$.

What are the BCs at the surface?

$$\nabla \times \underset{\sim}{H} = \underset{\sim}{J} + \epsilon_0 \frac{\partial}{\partial t} \underset{\sim}{E}$$

$\Rightarrow$ from Amperian loop with $\underset{\sim}{J}$, $\underset{\sim}{E}$ finite

$$\hat{n} \times \underset{\sim}{H} \Big|_{-}^{+} = 0$$

$\Rightarrow$ no singular currents for $\sigma \neq \infty$

From $\nabla \times \underset{\sim}{E} + \frac{\partial}{\partial t} \underset{\sim}{B} = 0$ and $\underset{\sim}{B}$ finite,

$$\hat{n} \times \underset{\sim}{E} \Big|_{-}^{+} = 0$$

From $\nabla \cdot \underset{\sim}{E} = \frac{\rho}{\epsilon_0} \Rightarrow \underset{\sim}{E} \cdot \hat{n} \Big|_{-}^{+} = \frac{\Sigma}{\epsilon_0}$

$\Rightarrow$ surface charge can be non-zero

From $\nabla \cdot \underset{\sim}{B} = 0 \Rightarrow \underset{\sim}{B} \cdot \hat{n} \Big|_{-}^{+} = 0$

Wave characteristics at a conducting surface

Outside:

$$\underset{\sim}{E}, \underset{\sim}{B} \sim e^{i\underset{\sim}{k}\cdot\underset{\sim}{x} - i\omega t}$$

$$k_0^2 = \frac{\omega^2}{c^2} = k_t^2 + k_{z0}^2$$

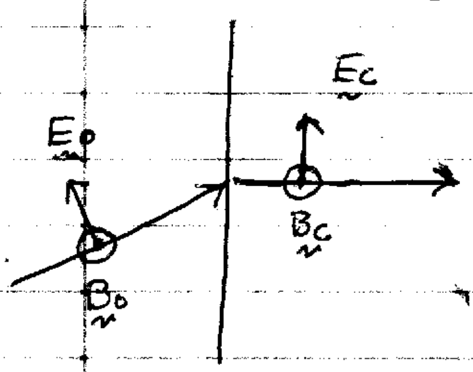
Inside:

$$k^2 = k_t^2 + k_{zc}^2$$

where $\quad k_{zc} = \dfrac{1+i}{\delta} \gg k_{z0}, k_t, \dfrac{\omega}{c}$

and k_t is unchanged across the boundary $\Rightarrow$ as discussed in the case of a dielectric interface

Therefore, inside $\underset{\sim}{k}_c \simeq k_{zc}\,\hat{z}$



Wave turns normal to the conducting surface.

Let $\quad \underset{\sim}{B}_c = B_c(0)\, e^{i\frac{z}{\delta}}\, e^{-\frac{z}{\delta}}$

Need to calculate $\underset{\sim}{E}_c$,

$$\nabla \times \underset{\sim}{B}_c = \mu_0 \underset{\sim}{J}_c = \mu_0 \sigma \underset{\sim}{E}_c$$

$$\Rightarrow \underset{\sim}{E}_c = \frac{1}{\mu_0\sigma}\nabla\times\underset{\sim}{B}_c = -\frac{1}{\sigma\mu_0}\hat{n}\times\underset{\sim}{B}_c\, k_{zc}$$

$$\underset{\sim}{E}_c = -\frac{1}{\sigma \mu_0}\frac{1}{\delta}(-1+i)\,\hat{n}\times \underset{\sim}{B}_c$$

Since $\sigma = \dfrac{2}{\mu_0 \omega \delta^2}$,

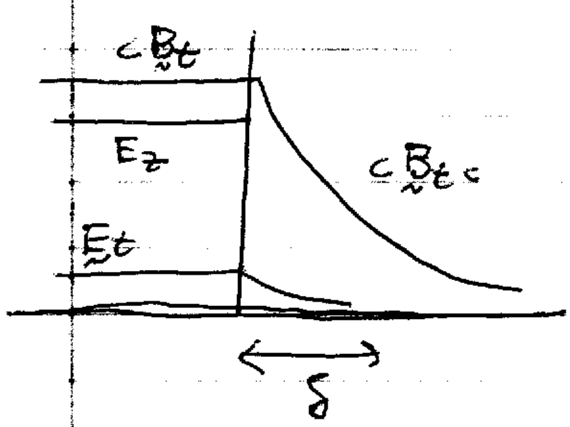
$$\underset{\sim}{E}_c = -\frac{\omega\delta}{2}(-1+i)\,\hat{n}\times \underset{\sim}{B}_c$$

Want to compare E_c / cB_c with the value in the vacuum where $E_0/cB_0 = 1$.

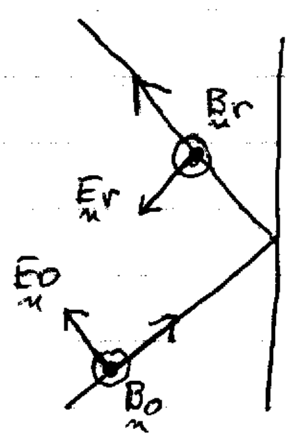
$$\frac{E_c}{cB_c} \simeq \frac{\omega \delta}{c} \sim k_0 \delta \ll 1$$

$\Rightarrow$ $\underset{\sim}{E}$ inside the conductor is small

$\Rightarrow$ But $\underset{\sim}{E}_t$ is contuous across the boundary so $\underset{\sim}{E}_t$ is also small in the vacuum region near the surface



Why is $\underset{\sim}{E}_t$ small in the vacuum?



Most wave energy reflected. Tangential $\underset{\sim}{E}_0$, $\underset{\sim}{E}_r$ cancel.

Is power flow into the conductor small?

$$\underset{\sim}{S} = \underset{\sim}{E} \times \underset{\sim}{H}$$

$$-\hat{n}\cdot\overline{\underset{\sim}{S}_c} = -\frac{1}{2}\,\mathrm{Re}\;\hat{n}\cdot\left(\underset{\sim}{E}_c \times B_c^*\right)\frac{1}{\mu_0}$$

$$= \frac{1}{2\mu_0}\,\mathrm{Re}\;\hat{n}\cdot\left(\underset{\sim}{B}_c^* \times \left(-\frac{\omega\delta}{2}\right)(i-1)\,\hat{n}\times\underset{\sim}{B}_c\right)$$

$$= -\frac{\omega\delta}{4\mu_0}\,\mathrm{Re}\left[(i-1)\,\hat{n}\cdot\underbrace{\left[\underset{\sim}{B}_c^* \times(\hat{n}\times\underset{\sim}{B}_c)\right]}_{|\underset{\sim}{B}_c|^2\hat{n} - \underset{\sim}{B}_c\,\underset{\sim}{B}_c^*\cdot\hat{n}}\right]$$

$$= +\frac{\omega\delta}{4\mu_0}\left(|\underset{\sim}{B}_c|^2 - |\hat{n}\cdot\underset{\sim}{B}_c|^2\right)$$

$$= \frac{\omega\delta}{4\mu_0}|\underset{\sim}{B}_t|^2 = \frac{k_0\delta}{4}\,c\,\frac{1}{\mu_0}|\underset{\sim}{B}_t|^2$$

$$\sim \frac{k_0\delta}{2}\left[c\,\frac{1}{2\mu_0}|\underset{\sim}{B}_t|^2\right] \sim \frac{k_0\delta}{2}\,S_0$$

$S_c \sim k_0\delta\, S_0 \ll S_0$ with S_0 the incident Poynting flux.

$\Rightarrow$ most of the wave energy is reflected.

$\Rightarrow$ $\hat{n}\times\underset{\sim}{E}$ is small