Magnetic fields and solids

We would like to develop a set of equations to describe the response of materials to magnetic fields. As in electrostatics, we consider a solid as having a large number of magnetic dipoles that respond to an imposed magnetic field.

$\Rightarrow$ as in electrostatics $\underset{\sim}{B}$ is the dipole component of the response to a localized current distribution

$\Rightarrow$ the response of materials is to produce a localized dipole moment per unit volume.

Define an average dipole moment per unit volume as

$$\underset{\sim}{M}(\underset{\sim}{x}) = \sum_i n_i \langle \underset{\sim}{m}_i \rangle$$

where $\langle \underset{\sim}{m}_i \rangle$ is the average magnetic moment of the ith class of molecules and n_i is the number density of that type of molecule. $\underset{\sim}{M}(\underset{\sim}{x})$ is an average over the molecular scale length but will generally vary over the macro-scale of physical objects.

We can calculate the vector potential $d\underset{\sim}{A}$ due to a small volume $d\underset{\sim}{x}'$ of an object as

$$d\underset{\sim}{A}(\underset{\sim}{x}) = \frac{\mu_0}{4\pi}\left[d\underset{\sim}{x}' \, \frac{\underset{\sim}{J}(\underset{\sim}{x}')}{|\underset{\sim}{x}-\underset{\sim}{x}'|} + \frac{\underset{\sim}{M}(\underset{\sim}{x}') \times (\underset{\sim}{x}-\underset{\sim}{x}')}{|\underset{\sim}{x}-\underset{\sim}{x}'|^3} \, d\underset{\sim}{x}' \right]$$

where $\underset{\sim}{J}(\underset{\sim}{x}')$ is the imposed macroscopic current density. Integrating over the volume,

$$\underset{\sim}{A}(\underset{\sim}{x}) = \frac{\mu_0}{4\pi}\int d\underset{\sim}{x}' \left[\frac{\underset{\sim}{J}(\underset{\sim}{x}')}{|\underset{\sim}{x}-\underset{\sim}{x}'|} + \underbrace{\frac{\underset{\sim}{M}(\underset{\sim}{x}') \times (\underset{\sim}{x}-\underset{\sim}{x}')}{|\underset{\sim}{x}-\underset{\sim}{x}'|^3}}_{\underset{\sim}{M}(\underset{\sim}{x}') \times \nabla' \frac{1}{|\underset{\sim}{x}-\underset{\sim}{x}'|}} \right]$$

Want to integrate by parts with respect to ∇'. Consider

$$\underbrace{\nabla' \times \frac{\underset{\sim}{M}(\underset{\sim}{x}')}{|\underset{\sim}{x}-\underset{\sim}{x}'|}} = \frac{1}{|\underset{\sim}{x}-\underset{\sim}{x}'|}\nabla' \times \underset{\sim}{M}(\underset{\sim}{x}') + \left(\nabla' \frac{1}{|\underset{\sim}{x}-\underset{\sim}{x}'|}\right) \times \underset{\sim}{M}(\underset{\sim}{x}')$$

integrates to
zero for $\underset{\sim}{M}$ localized.

$$\underset{\sim}{A}(\underset{\sim}{x}) = \frac{\mu_0}{4\pi}\int d\underset{\sim}{x}' \left[\frac{\underset{\sim}{J}(\underset{\sim}{x}')}{|\underset{\sim}{x}-\underset{\sim}{x}'|} + \frac{\nabla' \times \underset{\sim}{M}(\underset{\sim}{x}')}{|\underset{\sim}{x}-\underset{\sim}{x}'|} \right]$$

$\Rightarrow$ effective magnetization current density

$$\underset{\sim}{J}_M(\underset{\sim}{x}) = \nabla \times \underset{\sim}{M}(\underset{\sim}{x})$$

$\Rightarrow$ yields macroscopic differential equation

$$\nabla \times \underset{\sim}{B} = \mu_0 \left[\underset{\sim}{J} + \nabla \times \underset{\sim}{M} \right]$$

The total current consists of the macroscopic imposed current $\underset{\sim}{J}(\underset{\sim}{x})$ plus the self-generated magnetization current $\underset{\sim}{J}_M(\underset{\sim}{x})$.

We can define

$$\underset{\sim}{H} = \frac{1}{\mu_0}\underset{\sim}{B} - \underset{\sim}{M}$$

This yields the second eqn of magnetostatics

$$\nabla \times \underset{\sim}{H} = \underset{\sim}{J}$$

The field $\underset{\sim}{H}$ is a defined quantity that is produced only by the free currents $\underset{\sim}{J}$.

$\Rightarrow$ $\underset{\sim}{B}$ is the physical field that is measureable by evaluating forces acting on currents

$\Rightarrow$ $\underset{\sim}{B}$ is produced by all currents in a system.

The basic equations of magneto statics are then given by

$$\left.\begin{aligned} \nabla \times \underset{\sim}{H} &= \underset{\sim}{J} \\ \nabla \cdot \underset{\sim}{B} &= 0 \end{aligned}\right\} \qquad \begin{aligned} &\text{analagous to} \\ &\nabla \times \underset{\sim}{E} = 0 \\ &\nabla \cdot \underset{\sim}{D} = \rho \end{aligned}$$

with $\underset{\sim}{H} = \frac{1}{\mu_0}\underset{\sim}{B} - \underset{\sim}{M}$.

In many materials $\underset{\sim}{M}$ is proportional to $\underset{\sim}{B}$ and can write

$$\underset{\sim}{B} = \mu \underset{\sim}{H}$$

where μ is the magnetic permeability

Materials can be characterized by the ratio μ/μ_0.

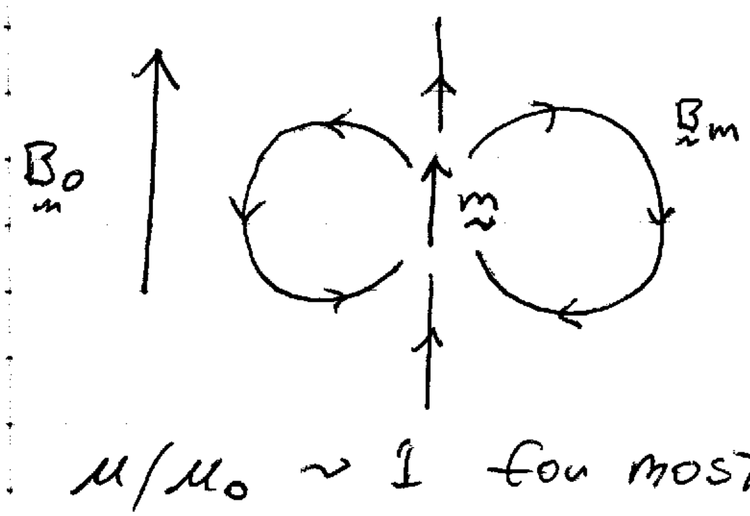
For $\mu/\mu_0 < 1 \Rightarrow$ diamagnetic

$\Rightarrow$ the material produces an $\underset{\sim}{M}$ that reduces $\underset{\sim}{B}$ within the material.

For $\mu/\mu_0 > 1 \Rightarrow$ paramagnetic

$\Rightarrow$ typically, the materials have intrinsic magnetic moments associated with unpaired electrons

$\Rightarrow$ magnetic dipoles align to reinforce $\underset{\sim}{B}$.

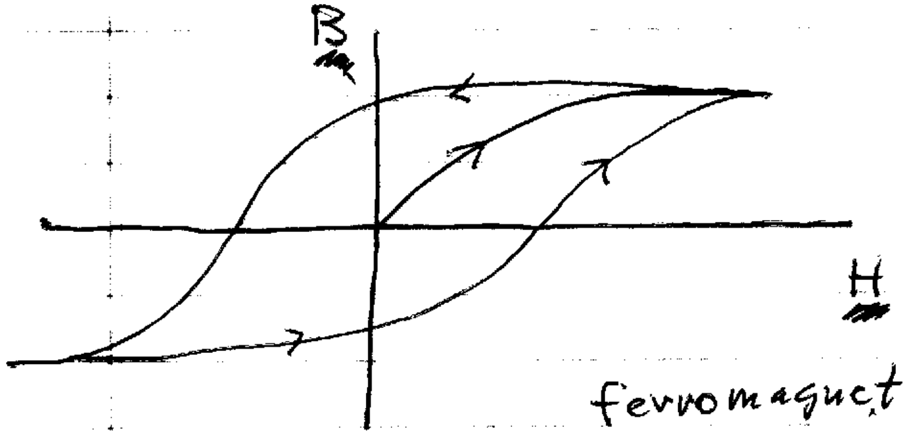


$\mu/\mu_0 \sim 1$ for most solids.

For ferromagnetic materials μ/μ_0 can be large and

$$\underset{\sim}{B} = \underset{\sim}{F}(\underset{\sim}{H})$$

where $\underset{\sim}{F}$ depends on the history of $\underset{\sim}{H}$.



Hysteresis

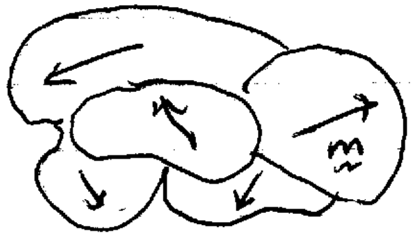
ferromagnetic

For $\underset{\sim}{H}$ small materials remain linear with

$$\underset{\sim}{B} = \mu \underset{\sim}{H}$$

but with μ/μ_0 large.

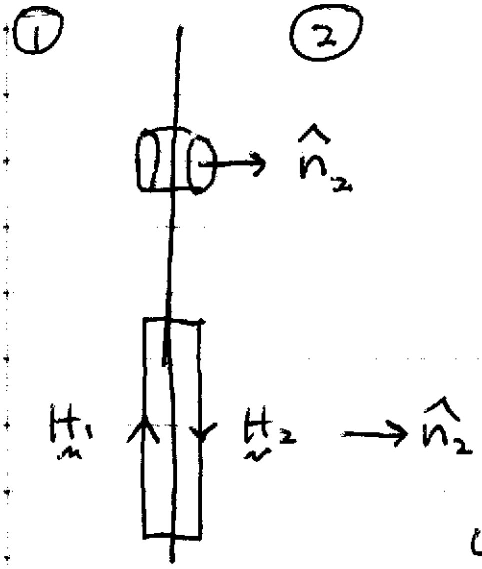
Ferromagnetic materials have intrinsic magnetic moments to are aligned in domains



The domains start to align as $\underset{\sim}{H}$ is imposed

$\Rightarrow$ large response

Boundary conditions for magnetic materials



From $\nabla \cdot \underset{\sim}{B} = 0$

$$(\underset{\sim}{B}_2 - \underset{\sim}{B}_1) \cdot \hat{n}_2 = 0$$

$$\hat{n}_2 \times (\underset{\sim}{H}_2 - \underset{\sim}{H}_1) = \text{surface current}$$

With no imposed surface current, $\underset{\sim}{H}$ tangent to the surface is continuous across the boundary.

Structure of $\underset{\sim}{B}$ at surfaces of high μ materials

From conservation of tangential H_t

$$H_t^{\mu} = H_t^{\mu_0}$$

Since $B = \mu H$

$$\frac{B_t^{\mu}}{\mu} = \frac{B_t^{\mu_0}}{\mu_0}$$

so $$B_t^{\mu} = \frac{\mu}{\mu_0} B_t^{\mu_0}$$

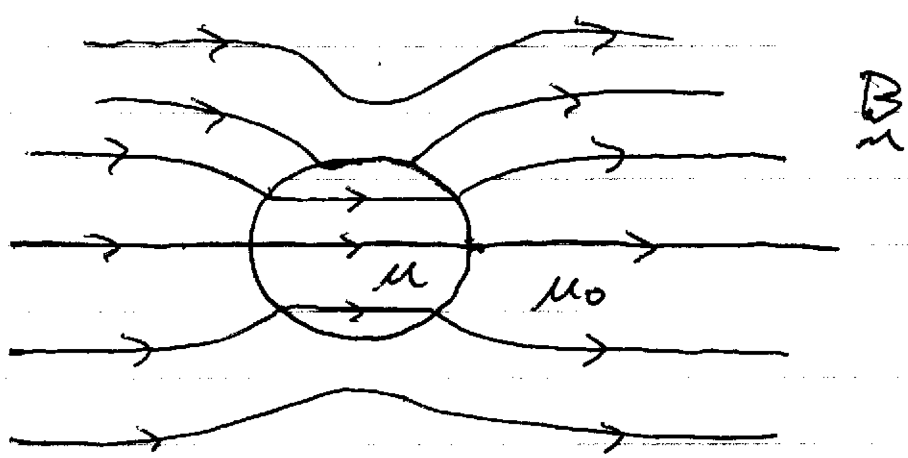
However, on physics grounds B_t^{μ} can not be very large $\Rightarrow$ not enough magnetic flux available

Thus, $B_t^{\mu_0}$ must be small.

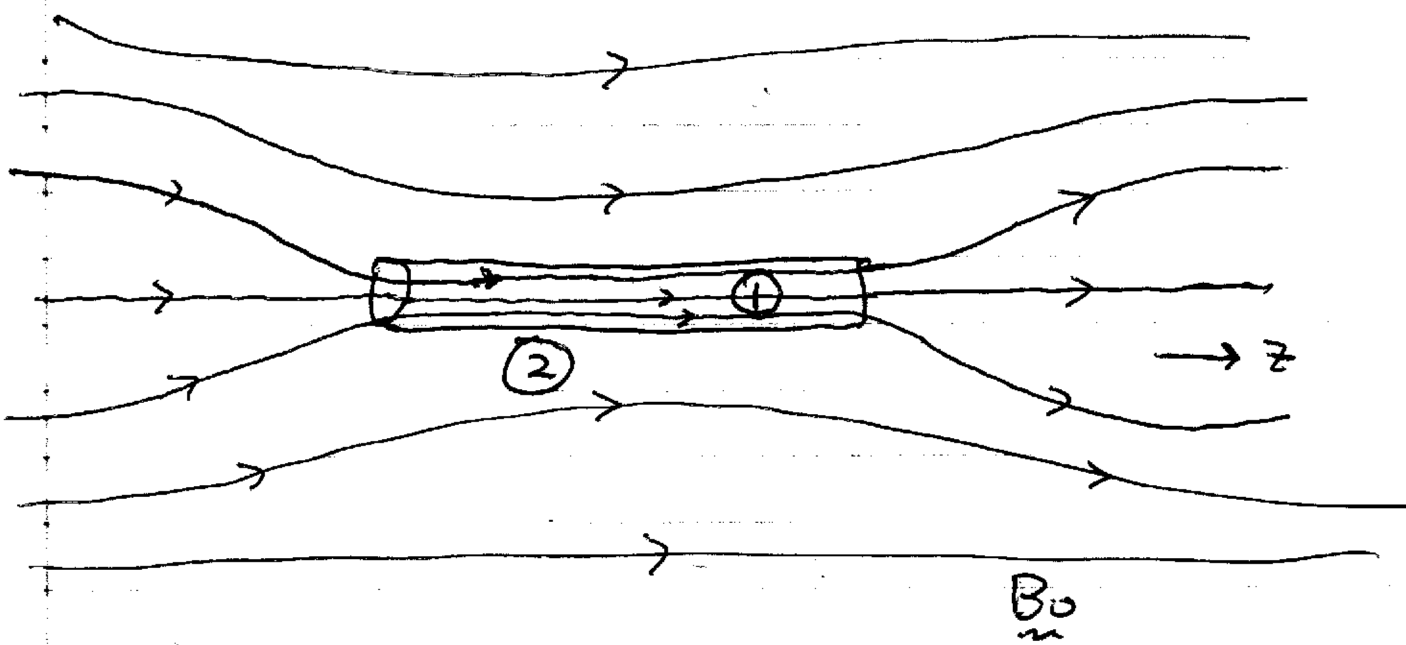
$\Rightarrow$ $B_t^{\mu_0}$ is small near the surface of high μ materials

$\Rightarrow$ $\underset{\sim}{B}$ is nearly normal to the surface

$\Rightarrow$ similar to $\underset{\sim}{E}$ near conductors



<u>Example</u> Scaling of B for an elongated cylinder with high μ. Length L and radius a. External field B_0



Since $\nabla \cdot \underset{\sim}{B} = 0$, magnetic flux is preserved. Across the surface

$$\frac{B_z^1}{\mu_1} = \frac{B_z^2}{\mu_0} \quad \text{from } H_t \text{ continuity}$$

$$\Rightarrow \quad B_z^1 = \frac{\mu_1}{\mu_0} B_z^2$$

High μ material pulls in magnetic flux from outside so B_z^2 is reduced below B_0.

Over what distance from the cylinder

is B_z reduced? Transverse scale must be L and not "a". An object changes its environment out to scales given by its largest dimension.

$\Rightarrow$ the ∇^2 operator in the system always links the scales in different directions

The initial magnetic flux due to B_0 over the region L is $B_0 \pi L^2$. The integrated magnetic flux in regions ① and ② must equal this flux.

$$B_0 \pi L^2 \simeq B_z^1 \pi a^2 + B_z^2 \pi L^2$$

$$= B_z^1 \left[\pi a^2 + \frac{\mu_0}{\mu_1} \pi L^2 \right]$$

$$\Rightarrow B_z^1 = B_0 \frac{L^2}{a^2} \frac{1}{1 + \frac{\mu_0}{\mu_1} \frac{L^2}{a^2}}$$

Note that for very large μ_1 / μ_0, the increase in B_z^1 is limited by the object size $\Rightarrow$ available magnetic flux

$$B_z^1 \simeq B_0 \frac{L^2}{a^2}$$

Solving problems with magnetic materials

Boundary value problems:

Since $\nabla \cdot \underset{\sim}{B} = 0 \implies \underset{\sim}{B} = \nabla \times \underset{\sim}{A}$

For $\underset{\sim}{B} = \mu \underset{\sim}{H}$

$$\nabla \times \left(\frac{1}{\mu} \nabla \times \underset{\sim}{A} \right) = \underset{\sim}{J}$$

If μ is piecewise constant and $\nabla \cdot \underset{\sim}{A} = 0$

$$\nabla^2 \underset{\sim}{A} = -\mu \underset{\sim}{J}$$

$\implies$ solve for $\underset{\sim}{A}$ in each region and use BCs to match the solutions across regions with different values of μ.

Systems with no free current: $\underset{\sim}{J} = 0$

$$\nabla \times \underset{\sim}{H} = 0 \implies \underset{\sim}{H} = -\nabla \varphi_m$$

$$\nabla \cdot \underset{\sim}{B} = -\nabla \cdot \mu \nabla \varphi_m = 0$$

For μ piecewise constant

$\nabla^2 \varphi_m = 0$ in each region

$\implies$ match solutions at interfaces.

Hard ferromagnetic materials:

$\Rightarrow$ $\underset{\sim}{M}$ fixed and $\underset{\sim}{J} = 0$, $\underset{\sim}{B} = \mu_0(\underset{\sim}{H}+\underset{\sim}{M})$

$\nabla \times \underset{\sim}{H} = 0 \Rightarrow \underset{\sim}{H} = -\nabla \Phi_m$

$\nabla \cdot \underset{\sim}{B} = 0 = \mu_0 \nabla \cdot (\underset{\sim}{H} + \underset{\sim}{M})$

$$\nabla^2 \Phi_m = \nabla \cdot \underset{\sim}{M} \equiv -\rho_m$$

$\Rightarrow$ calculate ρ_m

$\Rightarrow$ reduced to solving Poisson's equn.

When there are no boundaries

$$\Phi_m = -\frac{1}{4\pi}\int d\underset{\sim}{x}' \frac{\nabla' \cdot \underset{\sim}{M}(\underset{\sim}{x}')}{|\underset{\sim}{x} - \underset{\sim}{x}'|}$$

For $\underset{\sim}{M}$ localized integrate by parts so ∇' acts on $1/|\underset{\sim}{x} - \underset{\sim}{x}'|$. Then $\nabla' \Rightarrow -\nabla$

$$\Phi_m = -\frac{1}{4\pi}\nabla \cdot \int d\underset{\sim}{x}' \frac{\underset{\sim}{M}(\underset{\sim}{x}')}{|\underset{\sim}{x} - \underset{\sim}{x}'|}$$

For large $|\underset{\sim}{x}| \gg |\underset{\sim}{x}'|$

$$\Phi_m = \frac{1}{4\pi}\frac{\underset{\sim}{m} \cdot \underset{\sim}{x}}{|\underset{\sim}{x}|^3}, \quad \underset{\sim}{m} = \int d\underset{\sim}{x}' \underset{\sim}{M}(\underset{\sim}{x}')$$

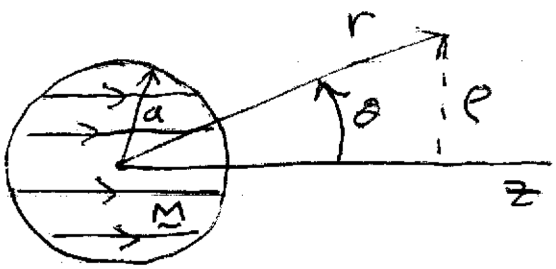
$$\underset{\sim}{H} = -\nabla \Phi_m$$

$\Rightarrow$ dipole field

example magnetized sphere

radius "a"

uniform $M = M\hat{z}$ for $r < a$.



$$z = r\cos\theta$$
$$r = (z^2+\rho^2)^{1/2}$$

since $J = 0 \Rightarrow$ no free current

$$\nabla\times H = 0 \Rightarrow H = -\nabla\varphi_m$$

$$B = \mu_0 H + \mu_0 M$$

$$\nabla\cdot B = 0 = \mu_0(\nabla\cdot H + \nabla\cdot M)$$

$$\nabla^2\varphi_m = \nabla\cdot M \quad , \quad M = \hat{z}\, M\, H(a-r)$$

$H = 1$ for positive argument
$\quad = 0$ for negative argument

$$\nabla\cdot M = M\frac{\partial}{\partial z}H = M\frac{\partial r}{\partial z}\underbrace{\frac{\partial H}{\partial r}}_{-\delta(a-r)}$$

$$\frac{\partial}{\partial z}r = \frac{\partial}{\partial z}(z^2+\rho^2)^{\frac{1}{2}}$$

$$= \frac{1}{2}\frac{1}{(z^2+\rho^2)^{1/2}}\,2z = \cos\theta$$

$$\nabla \cdot \underset{\sim}{M} = -M\cos\theta\ \delta(a-r)$$

$$\Phi_m = \sum_{\ell} P_\ell(\cos\theta)\, g_\ell(r) \quad \text{since } \frac{\partial}{\partial\varphi} = 0.$$

From $\nabla^2\Phi_m = \nabla\cdot\underset{\sim}{M} \Rightarrow$

$$\sum_{\ell}\left[\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} g_\ell - \ell(\ell+1)\frac{1}{r^2} g_\ell\right] P_\ell$$

$$= -M\cos\theta\ \delta(a-r)$$

Multiply by $P_{\ell'}$ and integrate $\cos\theta$ from -1 to 1.

$\Rightarrow$ elemmates sum over ℓ

$\Rightarrow$ recall that $\cos\theta = P_1(\cos\theta)$

$$\frac{2}{2\ell+1}\left[\frac{1}{r^2}\frac{\partial}{\partial r} r^2\frac{\partial}{\partial r} g_\ell - \ell(\ell+1)\frac{1}{r^2} g_\ell\right]$$

$$= -M\,\frac{2}{3}\,\delta_{\ell 1}\ \delta(a-r)$$

$\Rightarrow g_\ell = 0$ unless $\ell = 1$

$$\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} g_1 - 2\frac{g_1}{r^2} = -M\,\delta(a-r)$$

$\Rightarrow$ Euler eqn for $r \neq a$

$\Rightarrow g_1 \sim r^P$

$$P(P+1) - 2 = 0 \Rightarrow P = 1, -2$$

$$\frac{\partial^2}{\partial r^2} g_1 + \frac{2}{r}\frac{\partial}{\partial r} g_1 - \frac{2}{r^2} g_1 = -M\,\delta(a-r)$$

$\underline{r<a}$

$$g_1 = c_1 \frac{r}{a}$$

$\underline{r>a}$

$$g_1 = c_1 \left(\frac{a}{r}\right)^2$$

g_1 continuous at $r=a$

$\underline{r \simeq a}$

$$\frac{\partial^2}{\partial r^2} g_1 \simeq -M\,\delta(a-r)$$

$$\left.\frac{\partial g_1}{\partial r}\right|_{a-\epsilon}^{a+\epsilon} = -M$$

$$\left(-2c_1 \frac{a^2}{r^3} - c_1 \frac{1}{a}\right)_{r=a} = -M$$

$$-3c_1 \frac{1}{a} = -M \implies c_1 = \frac{Ma}{3}$$

$$\mathcal{Q}_m = \frac{M}{3} r\cos\theta = \frac{M}{3} z \quad r<a$$

$$= \underbrace{\frac{Ma^3}{3}\,\frac{\cos\theta}{r^2}}_{\dfrac{m\cos\theta}{4\pi r^2}} \quad r>a$$

for $m = \frac{4}{3}\pi a^3 M$ = magnetic moment

$$\underset{\sim}{B} = \mu_0 \left(-\nabla \varphi_m + \underset{\sim}{M} \right)$$

inside : $\underset{\sim}{B} = \mu_0 \left(-\frac{\underset{\sim}{M}}{3} + \underset{\sim}{M} \right) = \frac{2}{3} \mu_0 \underset{\sim}{M}$

$\Rightarrow$ const.

outside: etc

Alternate approach :

$\nabla^2 \varphi_m = 0$ everywhere but at the boundary at $r=a$.

Use the BCs on $\underset{\sim}{H}$ and $\underset{\sim}{B}$ at $r=a$.

$\Rightarrow$ tangential $\underset{\sim}{H}$ continuous

$$-\frac{1}{a} \frac{\partial}{\partial \theta} \varphi_m \Big|_{a-\epsilon}^{a+\epsilon} = 0$$

$$-\frac{1}{a} \left(\frac{\partial}{\partial \theta} \cos\theta \right) g_1(r) \Big|_{a-\epsilon}^{a+\epsilon} = 0$$

$$g_1(r) \Big|_{a-\epsilon}^{a+\epsilon} = 0$$

$\Rightarrow$ normal $\underset{\sim}{B}$ continuous

$$\mu_0 \left(H_r + M_r \right) \Big|_{a-\epsilon}^{a+\epsilon} = 0$$

$$H_r = -\frac{\partial \Phi_m}{\partial r}, \quad M_r = \cos\theta \, M \, H(a-r)$$

$$\cos\theta \left(-\frac{\partial}{\partial r} g_1 + M H(a-r) \right) \Bigg|_{a-\epsilon}^{a+\epsilon} = 0$$

$$-\frac{\partial}{\partial r} g_1 \Bigg|_{a-\epsilon}^{a+\epsilon} - M = 0$$

$$\frac{\partial g_1}{\partial r} \Bigg|_{a-\epsilon}^{a+\epsilon} = -M$$

$\Rightarrow$ same as from diff equ in r.