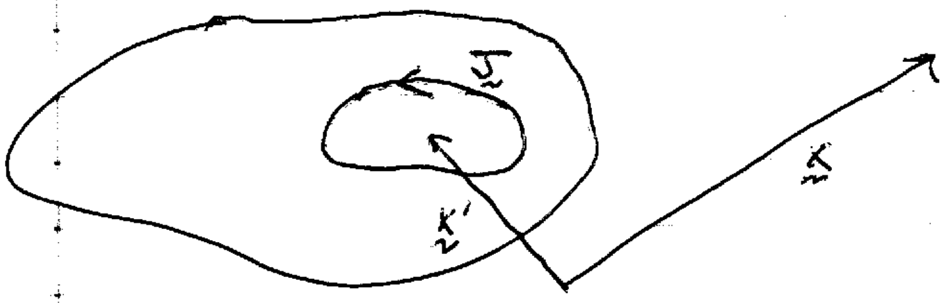
Magnetic fields from localized current distributions

As in the case of localized charge distributions, we want to be able to calculate the magnetic field from localized current distributions. This will also make it possible to calculate how materials respond to magnetic fields.



$$\underset{\sim}{A} = \frac{\mu_0}{4\pi} \int d\underset{\sim}{x}' \, \underset{\sim}{J}(\underset{\sim}{x}') \frac{1}{|\underset{\sim}{x}-\underset{\sim}{x}'|}$$

For $|\underset{\sim}{x}| \gg |\underset{\sim}{x}'|$

$$|\underset{\sim}{x}-\underset{\sim}{x}'| = \left(x^2 + x'^2 - 2\,\underset{\sim}{x}\cdot\underset{\sim}{x}'\right)^{\frac{1}{2}}$$

$$\simeq x\left(1 + \frac{x'^2}{x^2} - 2\frac{\underset{\sim}{x}\cdot\underset{\sim}{x}'}{x^2}\right)^{1/2}$$

$$\simeq x\left(1 - \frac{\underset{\sim}{x}\cdot\underset{\sim}{x}'}{x^2} + \cdots\right)$$

$$\frac{1}{|\underset{\sim}{x}-\underset{\sim}{x}'|} \simeq \frac{1}{x} + \frac{\underset{\sim}{x}\cdot\underset{\sim}{x}'}{x^3} + \cdots\cdots$$

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int d\mathbf{x}' \, \mathbf{J}(\mathbf{x}') \left[\frac{1}{x} + \frac{\mathbf{x}\cdot\mathbf{x}'}{x^3} + \cdots \right]$$

$$\int d\mathbf{x}' \, \nabla'\cdot\left(\mathbf{J}(\mathbf{x}')\,\mathbf{x}'\right) = 0 \quad \text{since } J \to 0 \text{ at } \infty$$

$$= \int d\mathbf{x}' \left[\underbrace{\left(\nabla'\cdot\mathbf{J}(\mathbf{x}')\right)}_{\substack{=0 \text{ since} \\ \nabla'\cdot\mathbf{J}=0 \\ \text{for } \frac{\partial}{\partial t}=0}} \mathbf{x}' + \underbrace{\mathbf{J}(\mathbf{x}')\cdot\nabla'\mathbf{x}'}_{\mathbf{J}(\mathbf{x}')\cdot \mathbf{I}} \right]$$

$\mathbf{I}$ = unit tensor

$$\Rightarrow \int d\mathbf{x}' \, \mathbf{J}(\mathbf{x}') = 0$$

$\Rightarrow$ no monopole contributions to $\mathbf{A}$

$$\mathbf{A} = \frac{\mu_0}{4\pi} \frac{\mathbf{x}\cdot}{x^3} \int d\mathbf{x}' \, \mathbf{x}' \, \mathbf{J}(\mathbf{x}')$$

$$\mathbf{x} \times (\mathbf{x}' \times \mathbf{J}) = \mathbf{x}\cdot\underbrace{\mathbf{J}(\mathbf{x}')\,\mathbf{x}'} - \mathbf{x}\cdot\mathbf{x}' \, \mathbf{J}(\mathbf{x}')$$

$$\int d\mathbf{x}' \, \mathbf{J}(\mathbf{x}')\,\mathbf{x}' = \int d\mathbf{x}' \left(\mathbf{J}\cdot\underbrace{\nabla'\mathbf{x}'}_{\mathbf{I}}\right)\mathbf{x}'$$

$$= -\int d\mathbf{x}' \, \mathbf{x}' \, \nabla'\cdot(\mathbf{J}\,\mathbf{x}')$$

$$= -\int d\mathbf{x}'\, \mathbf{x}'\, \mathbf{J}\cdot\nabla'\mathbf{x}' = -\int d\mathbf{x}'\, \mathbf{x}'\, \mathbf{J}\cdot\mathbf{I}$$

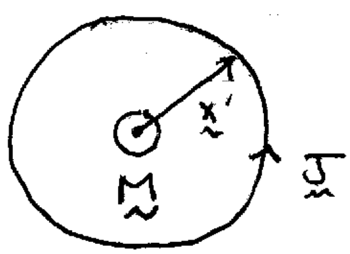
$$= -\int d\mathbf{x}'\, \mathbf{x}'\, \mathbf{J}(\mathbf{x}')$$

$$\Rightarrow \int d\mathbf{x}'\, \mathbf{x}\times\left[\mathbf{x}'\times\mathbf{J}(\mathbf{x}')\right] = -2\int d\mathbf{x}'\, \mathbf{x}\cdot\mathbf{x}'\, \mathbf{J}(\mathbf{x}')$$

$$\mathbf{A} = -\frac{1}{2}\,\frac{\mu_0}{4\pi}\,\frac{1}{x^3}\int d\mathbf{x}'\, \mathbf{x}\times\left[\mathbf{x}'\times\mathbf{J}(\mathbf{x}')\right]$$

Define $\mathbf{M}(\mathbf{x}') = \frac{1}{2}\,\mathbf{x}'\times\mathbf{J}(\mathbf{x}')$

= magnetic moment density



$$\mathbf{m} = \int d\mathbf{x}'\, \mathbf{M}(\mathbf{x}') = \text{magnetic moment}$$

$$\sim L^3\left(L\,\frac{\text{amps}}{L^2}\right) \sim \text{amps}\, L^2$$

$$\sim I\ \text{area}$$

$$\mathbf{A} = -\frac{\mu_0}{4\pi}\,\frac{1}{x^3}\,\mathbf{x}\times\mathbf{m}$$

$$\underset{\sim}{B} = \nabla \times \underset{\sim}{A} = -\frac{\mu_0}{4\pi} \nabla \times \left(\frac{\underset{\sim}{x} \times \underset{\sim}{m}}{x^3} \right)$$

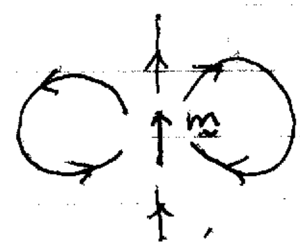
$$= -\frac{\mu_0}{4\pi} \left[\underset{\sim}{m} \cdot \nabla \left(\frac{\underset{\sim}{x}}{x^3} \right) - \underset{\sim}{m} \, \underbrace{\nabla \cdot \left(\frac{\underset{\sim}{x}}{x^3} \right)}_{\underbrace{-\nabla\left(\frac{1}{x}\right)}_{-\nabla^2 \frac{1}{x} = 4\pi \delta(\underset{\sim}{x}) = 0}} \right]$$

$$= -\frac{\mu_0}{4\pi} \underset{\sim}{m} \cdot \nabla \left(\frac{\underset{\sim}{x}}{x^3} \right)$$

$$= -\frac{\mu_0}{4\pi} \left[\frac{\underset{\sim}{m} \cdot \underset{\approx}{I}}{x^3} - \underset{\sim}{x} \, \underset{\sim}{m} \cdot \nabla \frac{1}{x^3} \right]$$

$$\nabla \frac{1}{x^3} = -\frac{3}{2} \frac{1}{x^5} \, 2 \underset{\sim}{x} = -3 \frac{\underset{\sim}{x}}{x^5}$$

$$\underset{\sim}{B} = \frac{\mu_0}{4\pi} \left(\frac{3 \hat{x} \, \hat{x} \cdot \underset{\sim}{m} - \underset{\sim}{m}}{x^3} \right)$$



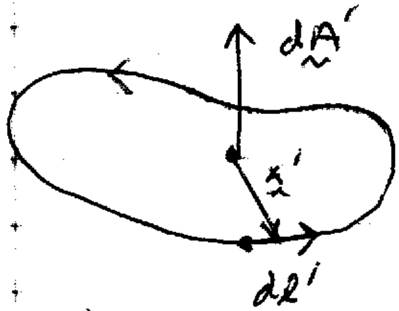
$\Rightarrow$ same form as electric dipole

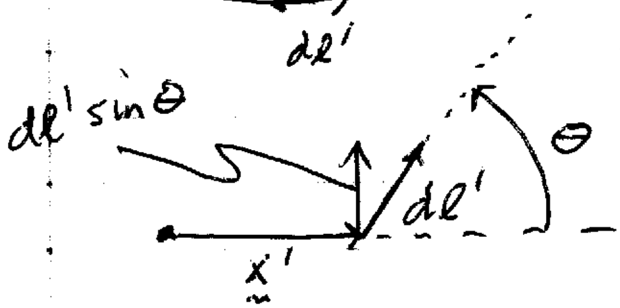
$\Rightarrow$ magnetic field far from a localized current distribution is to lowest order dipolar.

Magnetic moment from a plane current loop

$$\underset{\sim}{m} = \frac{1}{2}\int d\underset{\sim}{x}'\ \underset{\sim}{x}' \times \underset{\sim}{J}(\underset{\sim}{x}')$$

$$= \frac{1}{2} I \oint \underset{\sim}{x}' \times d\underset{\sim}{l}'$$





$$\frac{1}{2}\underset{\sim}{x}' \times d\underset{\sim}{l}' = x' \frac{dl' \sin\theta}{2}$$

$$= dA'$$

$=$ area of segment

$$\underset{\sim}{m} = I \int d\underset{\sim}{A}' = I \underset{\sim}{A}$$

A = area of loop

$\Rightarrow$ direction of $\underset{\sim}{A}$ given by right hand rule.

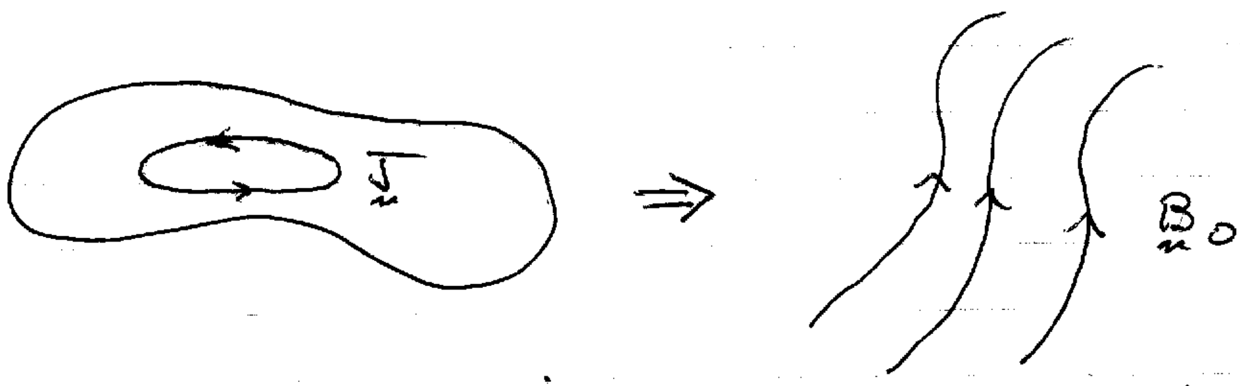
$\Rightarrow$ generalizes result from circular current loop where found

$$m = I \pi a^2$$

$\Rightarrow$ can extend to quadrupole moments

Magnetic forces

We previously calculated the energy of a charge distribution in an external electric field $\mathbf{E}_0$ and the calculated the forces using virtual displacements. We can, in principle, do the same for currents



However, energy is required to maintain the currents $\mathbf{J}$ because magnetic flux cuts through the current loops

$\Rightarrow$ need Faraday's law which we have not yet discussed

$\Rightarrow$ Instead calculate the forces directly.

$$\mathbf{F} = \int d\mathbf{x}' \; \mathbf{J}(\mathbf{x}') \times \mathbf{B}(\mathbf{x}')$$

where $\mathbf{B}$ is the external field and not the field produced by $\mathbf{J}$

$\Rightarrow$ $\mathbf{J}$ can not accelerate itself

Assume that $\mathbf{B}$ is slowly varying over the spatial scale of $\mathbf{J}$.

$$\mathbf{B}(\mathbf{x}') \cong \mathbf{B}(\mathbf{x}) + (\mathbf{x}'-\mathbf{x})\cdot\nabla\,\mathbf{B}(\mathbf{x}) + \cdots$$

$$\mathbf{F} = \int d\mathbf{x}'\; \mathbf{J}(\mathbf{x}') \times \mathbf{B}(\mathbf{x}) + \int d\mathbf{x}'\; \mathbf{J}(\mathbf{x}') \times \left[(\mathbf{x}'-\mathbf{x})\cdot\nabla\,\mathbf{B}\right]$$

$$= \int d\mathbf{x}'\; \left[\mathbf{J}(\mathbf{x}')\; \mathbf{x}'\cdot\nabla\right] \times \mathbf{B}(\mathbf{x})$$

$$\int d\mathbf{x}'\; \nabla\times\left[\mathbf{x}'\times\mathbf{J}(\mathbf{x}')\right] = \int d\mathbf{x}'\left[\,\mathbf{x}'\;\mathbf{J}\cdot\nabla - \mathbf{J}\;\mathbf{x}'\cdot\nabla\right]$$

$\Rightarrow$ note that this is an operator equ where ∇ does not act on $\mathbf{x}'$ or $\mathbf{J}(\mathbf{x})$.

$$= -2\int d\mathbf{x}'\; \mathbf{J}(\mathbf{x}')\; \mathbf{x}'\cdot\nabla$$

where $\mathbf{x}'\,\mathbf{J}(\mathbf{x}') \Rightarrow -\mathbf{J}(\mathbf{x}')\,\mathbf{x}'$ as before

$$\underset{\sim}{F} = -\frac{1}{2} \int d\underset{\sim}{x}' \left(\nabla \times \left[\underset{\sim}{x}' \times \underset{\sim}{J}(\underset{\sim}{x}') \right] \right) \times \underset{\sim}{B}(\underset{\sim}{x})$$

$$= -\left(\nabla \times \underset{\sim}{m} \right) \times \underset{\sim}{B} = \nabla \left(\underset{\sim}{m} \cdot \underset{\sim}{B} \right) - \cancel{\underset{\sim}{m}\, \nabla \cdot \underset{\sim}{B}}^{\,0}$$

$$\underset{\sim}{F} = \nabla \left(\underset{\sim}{m} \cdot \underset{\sim}{B} \right)$$

$\Rightarrow$ where ∇ acts on $\underset{\sim}{B}(\underset{\sim}{x})$ only

$\Rightarrow$ $\underset{\sim}{m}$ is the magnetic moment of the current distribution

Can write

$$U = -\underset{\sim}{m} \cdot \underset{\sim}{B}$$

as the energy of $\underset{\sim}{m}$ in an external field $\underset{\sim}{B}$ with

$$\underset{\sim}{F} = -\nabla U$$

Torque on a current distribution

Consider the torque $d\underset{\sim}{N}' = \underset{\sim}{x}' \times d\underset{\sim}{F}'$ on a volume element $d\underset{\sim}{x}'$

$$\underset{\sim}{N} = \int d\underset{\sim}{N}' = \int d\underset{\sim}{x}' \; \underset{\sim}{x}' \times \left(\underset{\sim}{J}(\underset{\sim}{x}') \times \underset{\sim}{B}(\underset{\sim}{x}') \right)$$

To lowest order $\underset{\sim}{B}(\underset{\sim}{x}') = \underset{\sim}{B}(0)$ where take the origin of $\underset{\sim}{J}$ to be at zero.

$$\underset{\sim}{N} = \int d\underset{\sim}{x}' \; \underset{\sim}{x}' \times \left[\underset{\sim}{J}(\underset{\sim}{x}') \times \underset{\sim}{B}(0)\right]$$

$$= \int d\underset{\sim}{x}' \left[\underset{\sim}{B} \cdot \underset{\sim}{x}' \, \underset{\sim}{J}(\underset{\sim}{x}') - \underset{\sim}{B} \; \underset{\sim}{x}' \cdot \underset{\sim}{J}(\underset{\sim}{x}') \right]$$

$$\int d\underset{\sim}{x}' \; \underset{\sim}{x}' \cdot \underset{\sim}{J}(\underset{\sim}{x}') = \int d\underset{\sim}{x}' \; \underset{\sim}{J}(\underset{\sim}{x}') \cdot \nabla' \left(\frac{\underset{\sim}{x}' \cdot \underset{\sim}{x}'}{2} \right)$$

$$= - \int d\underset{\sim}{x}' \; \frac{x'^2}{2} \, \nabla' \cdot \underset{\sim}{J}(\underset{\sim}{x}') = 0$$

in steady state.

$$N = \underset{\sim}{B} \cdot \int dx' \; \underset{\sim}{x}' \, \underset{\sim}{J}(\underset{\sim}{x}')$$

Have

$$\underset{\sim}{B} \times \left[\underset{\sim}{x}' \times \underset{\sim}{J}(\underset{\sim}{x}') \right] = \underset{\sim}{x}' \; \underset{\sim}{J}(\underset{\sim}{x}') \cdot \underset{\sim}{B}$$

$$- \underset{\sim}{J}(\underset{\sim}{x}') \; \underset{\sim}{x}' \cdot \underset{\sim}{B}$$

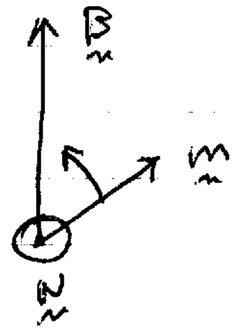
$$\Rightarrow - 2 \; \underset{\sim}{J}(\underset{\sim}{x}') \; \underset{\sim}{x}' \cdot \underset{\sim}{B}$$

where again $\underset{\sim}{x}' \underset{\sim}{J} \Rightarrow - \underset{\sim}{J} \underset{\sim}{x}'$
when integrated over $\underset{\sim}{x}'$.

$$\Rightarrow \quad \mathbf{B} \cdot \mathbf{x}' \, \mathbf{J}(\mathbf{x}') \Rightarrow -\frac{1}{2} \mathbf{B} \times \left[\mathbf{x}' \times \mathbf{J}(\mathbf{x}') \right]$$

$$\mathbf{N} = -\frac{1}{2} \mathbf{B} \times \int d\mathbf{x}' \left[\mathbf{x}' \times \mathbf{J}(\mathbf{x}') \right] = -\mathbf{B} \times \mathbf{m}$$

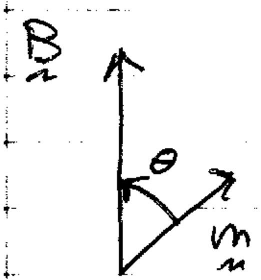
$$\mathbf{N} = \mathbf{m} \times \mathbf{B}$$



$\mathbf{m}$ twists to align with $\mathbf{B}$

$\Rightarrow$ consistent with

$$U = -\mathbf{m} \cdot \mathbf{B} = -mB\cos\theta$$



$$N = -\frac{\partial}{\partial\theta} U = mB\sin\theta$$