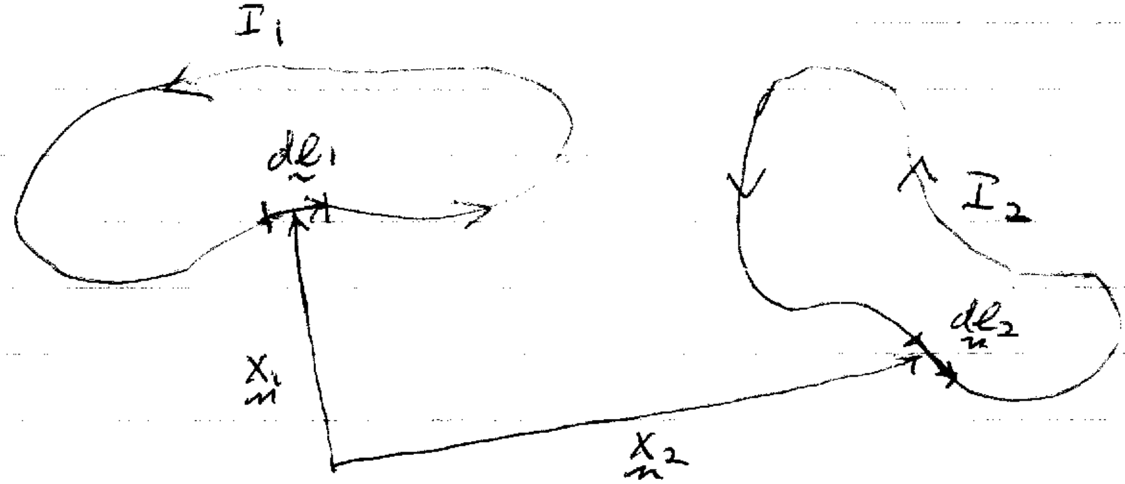
Magnetostatics

Describes the interactions between currents in a system without time dependence.

Observations revealed the forces between current carrying wires.



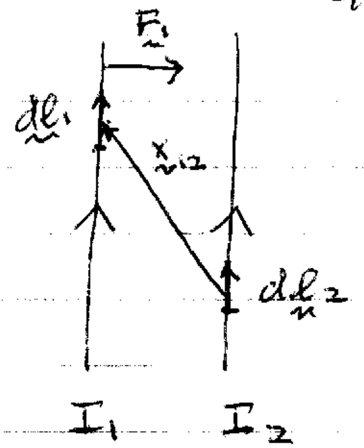
$$\frac{dI_1}{dt} = \frac{dI_2}{dt} = 0 \qquad \frac{\mu_0}{4\pi} = 10^{-7} \frac{N}{A^2}$$

SI units

$$\underset{\sim}{F}_1 = \frac{\mu_0}{4\pi} I_1 I_2 \oint\oint d\underset{\sim}{l}_1 \times \frac{\left[d\underset{\sim}{l}_2 \times (\underset{\sim}{x}_1 - \underset{\sim}{x}_2) \right]}{|\underset{\sim}{x}_1 - \underset{\sim}{x}_2|^3}$$

	cgs	MKS/SI
F	dynes	Newtons
I	statcoulombs/s	Amperes = Coulomb/s
X	cm	m

For cgs $\frac{\mu_0}{4\pi} \longrightarrow \frac{1}{c^2}$



Parallel wires attract

Need to show that $\underset{\sim}{F}_1 = -\underset{\sim}{F}_2$

$\Rightarrow$ required for momentum conservation

$\Rightarrow$ expand the cross products in $\underset{\sim}{F}_1$

$$\underset{\sim}{F}_1 = \frac{\mu_0}{4\pi} I_1 I_2 \oint\oint \left[\frac{d\underset{\sim}{l}_2\, d\underset{\sim}{l}_1 \cdot (\underset{\sim}{x}_1 - \underset{\sim}{x}_2) - d\underset{\sim}{l}_1 \cdot d\underset{\sim}{l}_2 (\underset{\sim}{x}_1 - \underset{\sim}{x}_2)}{|\underset{\sim}{x}_1 - \underset{\sim}{x}_2|^3} \right]$$

but
$$\frac{\underset{\sim}{x}_1 - \underset{\sim}{x}_2}{|\underset{\sim}{x}_1 - \underset{\sim}{x}_2|^3} = -\nabla_1 \frac{1}{|\underset{\sim}{x}_1 - \underset{\sim}{x}_2|}$$

$$\oint\oint \frac{d\underset{\sim}{l}_2\, d\underset{\sim}{l}_1 \cdot (\underset{\sim}{x}_1 - \underset{\sim}{x}_2)}{|\underset{\sim}{x}_1 - \underset{\sim}{x}_2|^3} = -\oint\oint d\underset{\sim}{l}_2 \underbrace{d\underset{\sim}{l}_1 \cdot \nabla_1 \frac{1}{|\underset{\sim}{x}_1 - \underset{\sim}{x}_2|}}$$

Integral over $d\underset{\sim}{l}_1$, a perfect differential $= 0$

$$\underset{\sim}{F}_1 = -\frac{\mu_0}{4\pi} I_1 I_2 \oint\oint d\underset{\sim}{l}_1 \cdot d\underset{\sim}{l}_2 \frac{(\underset{\sim}{x}_1 - \underset{\sim}{x}_2)}{|x_1 - x_2|^3}$$

$$= -\underset{\sim}{F}_2$$

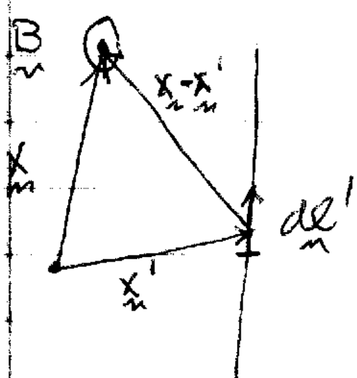
Can define a force field $\underset{\sim}{B}$ associated with the current I,

$$\underset{\sim}{B}(\underset{\sim}{x}) = \frac{\mu_0}{4\pi} I \oint d\underset{\sim}{l}' \times (\underset{\sim}{x} - \underset{\sim}{x}') \frac{1}{|\underset{\sim}{x} - \underset{\sim}{x}'|^3}$$

$\underset{\sim}{B}$ is the magnetic induction

	cgs	SI
B	Gauss	Tesla $= 10^4$ G

Direction of $\underset{\sim}{B}$ given by the right hand rule



The force is given by

$$\underset{\sim}{F}_1 = I_1 \int d\underset{\sim}{l}_1 \times \underset{\sim}{B}(\underset{\sim}{x}_1)$$

Generalization to continuous current distributions

$$\int d\underset{\sim}{l}' \, I \Rightarrow \int d\underset{\sim}{x}' \, \underset{\sim}{J}(\underset{\sim}{x}')$$

$\Rightarrow \underset{\sim}{J}$ is the current per unit area

$$\underset{\sim}{B}(\underset{\sim}{x}) = \frac{\mu_0}{4\pi} \int d\underset{\sim}{x}' \, \frac{\underset{\sim}{J}(\underset{\sim}{x}') \times (\underset{\sim}{x} - \underset{\sim}{x}')}{|\underset{\sim}{x} - \underset{\sim}{x}'|^3}$$

$$= -\frac{\mu_0}{4\pi} \int d\underset{\sim}{x}' \, \underbrace{\underset{\sim}{J}(\underset{\sim}{x}') \times \nabla \frac{1}{|\underset{\sim}{x} - \underset{\sim}{x}'|}}_{-\nabla \times \frac{\underset{\sim}{J}(\underset{\sim}{x}')}{|\underset{\sim}{x} - \underset{\sim}{x}'|}}$$

$$= \frac{\mu_0}{4\pi} \nabla \times \int d\underset{\sim}{x}' \, \frac{\underset{\sim}{J}(\underset{\sim}{x}')}{|\underset{\sim}{x} - \underset{\sim}{x}'|}$$

since ∇ acts on $\underset{\sim}{x}$ and not $\underset{\sim}{x}'$.

$\Rightarrow \quad \nabla \cdot \underset{\sim}{B} = 0 \quad$ First law of magnetostatics

$\Rightarrow$ no magnetic monopoles

Still need a differential equation for $\underset{\sim}{B}$ driven by $\underset{\sim}{J}$. Take the curl of $\underset{\sim}{B}$,

$$\nabla \times \underset{\sim}{B} = \nabla \times \{ \nabla \times [\ ] \}$$

$$[\ ] = \frac{\mu_0}{4\pi} \int d\underset{\sim}{x}' \, \underset{\sim}{J}(\underset{\sim}{x}') \frac{1}{|\underset{\sim}{x} - \underset{\sim}{x}'|}$$

$$\nabla \times \underset{\sim}{B} = \nabla\, \nabla \cdot [\;] - \nabla^2 [\;]$$

$$\nabla \cdot [\;] = \nabla \cdot \frac{\mu_0}{4\pi} \int d\underset{\sim}{x}'\; \underset{\sim}{J}(\underset{\sim}{x}') \frac{1}{|\underset{\sim}{x} - \underset{\sim}{x}'|}$$

$$= \frac{\mu_0}{4\pi} \int d\underset{\sim}{x}'\; \underset{\sim}{J}(\underset{\sim}{x}') \cdot \underbrace{\nabla \frac{1}{|\underset{\sim}{x} - \underset{\sim}{x}'|}}_{-\nabla' \frac{1}{|\underset{\sim}{x} - \underset{\sim}{x}'|}}$$

$$\nabla' \cdot \left[\underset{\sim}{J}(\underset{\sim}{x}') \frac{1}{|\underset{\sim}{x} - \underset{\sim}{x}'|} \right] = \frac{1}{|\underset{\sim}{x} - \underset{\sim}{x}'|} \nabla' \cdot \underset{\sim}{J}(\underset{\sim}{x}') + \underset{\sim}{J}(\underset{\sim}{x}') \cdot \nabla' \frac{1}{|\underset{\sim}{x} - \underset{\sim}{x}'|}$$

$$\nabla \cdot [\;] = -\frac{\mu_0}{4\pi} \int d\underset{\sim}{x}' \left[\nabla' \cdot \left[\underset{\sim}{J}(\underset{\sim}{x}') \frac{1}{|\underset{\sim}{x} - \underset{\sim}{x}'|} \right] - \frac{1}{|\underset{\sim}{x} - \underset{\sim}{x}'|} \nabla' \cdot \underset{\sim}{J}(\underset{\sim}{x}') \right]$$

From divergences theorem and zero BC at infinity

$$\nabla \cdot [\;] = \frac{\mu_0}{4\pi} \int d\underset{\sim}{x}' \frac{1}{|\underset{\sim}{x} - \underset{\sim}{x}'|} \nabla' \cdot \underset{\sim}{J}(\underset{\sim}{x}')$$

Continuity of charge requires

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \underset{\sim}{J} = 0 .$$

In steady state $\partial\rho/\partial t = 0$

$$\Rightarrow \nabla \cdot \underset{\sim}{J} = 0$$

Thus, $\nabla \cdot [\,] = 0$.

$$\nabla^2 [\,] = \frac{\mu_0}{4\pi} \int d\underset{\sim}{x}' \, \underset{\sim}{J}(\underset{\sim}{x}') \underbrace{\nabla^2 \frac{1}{|\underset{\sim}{x} - \underset{\sim}{x}'|}}_{-4\pi\,\delta(\underset{\sim}{x} - \underset{\sim}{x}')}$$

$$= \frac{\mu_0}{4\pi} \int d\underset{\sim}{x}' \, \underset{\sim}{J}(\underset{\sim}{x}') \left[-4\pi\,\delta(\underset{\sim}{x} - \underset{\sim}{x}') \right]$$

$$= -\mu_0 \, \underset{\sim}{J}(\underset{\sim}{x})$$

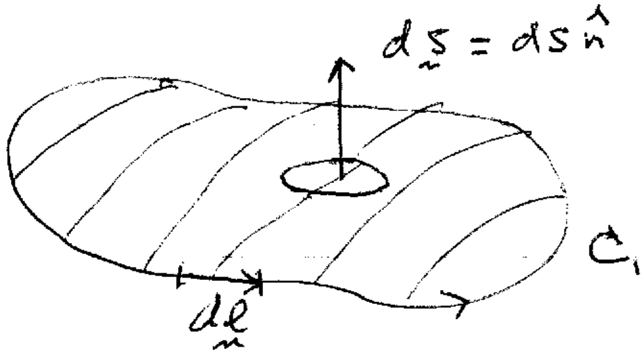
So,

$$\nabla \times \underset{\sim}{B} = \mu_0 \, \underset{\sim}{J}(\underset{\sim}{x})$$ Second law of magnetostatics.

Ampere's Law

Want to obtain an integral representation for $\underset{\sim}{B}$ that parallels Gauss' law for electric fields.

Consider a surface $\underset{\sim}{S}$ bounded by a curve C,



Direction of $d\underset{\sim}{S}$ given by the right hand rule

Consider

$$\int_S d\underset{\sim}{S} \cdot \nabla \times \underset{\sim}{B} = \mu_0 \int_S d\underset{\sim}{S} \cdot \underset{\sim}{J}(\underset{\sim}{x})$$

$$= \mu_0 I,$$

where I is the total current passing through S.

$$I = \int_S d\underset{\sim}{S} \cdot \underset{\sim}{J}$$

Using Stokes theorem

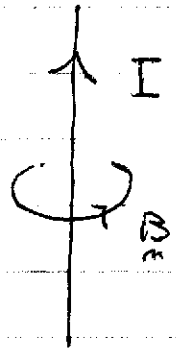
$$\int d\underset{\sim}{S} \cdot \nabla \times \underset{\sim}{B} = \oint_C d\underset{\sim}{\ell} \cdot \underset{\sim}{B}$$

$\Rightarrow \quad \oint \underset{\sim}{B} \cdot d\underset{\sim}{\ell} = \mu_0 I$ Ampere's law

$\Rightarrow$ always valid but only useful in systems with symmetry.

Example:

Magnetic field from current carrying wire



$$\oint \mathbf{B} \cdot d\mathbf{l} = B\, 2\pi r = \mu_0 I$$

$$B = \frac{\mu_0 I}{2\pi r}$$

B from right hand rule

Vector Potential

Since $\nabla \cdot \mathbf{B} = 0$, can write

$$\mathbf{B} = \nabla \times \mathbf{A}$$

with $\mathbf{A}$ the vector potential. Since

$$\mathbf{B} = \nabla \times \frac{\mu_0}{4\pi} \int d\mathbf{x}'\, \mathbf{J}(\mathbf{x}') \frac{1}{|\mathbf{x} - \mathbf{x}'|},$$

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int d\mathbf{x}' \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} + \nabla \psi$$

This result is analagous to the expression for Φ as an integral over the charge ρ in an infinite medium.

The scalar ψ is an arbitrary function since

$$\underset{\sim}{B} = \nabla \times \underset{\sim}{A}$$

is independent of ψ. Thus, $\underset{\sim}{A}$ is not unique. Consider the transformation

$$\underset{\sim}{A} \Longrightarrow \underset{\sim}{A}' + \nabla \psi$$

Guage transformation

This leaves $\underset{\sim}{B}$ unchanged. The guage transformation allows us to choose $\nabla \cdot \underset{\sim}{A}$ to be whatever we want

$$\nabla \cdot \underset{\sim}{A} = \nabla \cdot A' + \nabla^2 \psi$$

Choosing $\nabla^2 \psi = \nabla \cdot \underset{\sim}{A}$ gives $\nabla \cdot \underset{\sim}{A}' = 0$.
$\nabla \cdot \underset{\sim}{A} = 0$ is the Coulomb guage.

From $\qquad \nabla \times \underset{\sim}{B} = \mu_0 \underset{\sim}{J}$

$$\nabla \times (\nabla \times \underset{\sim}{A}) = \nabla (\nabla \cdot \underset{\sim}{A}) - \nabla^2 \underset{\sim}{A} = \mu_0 \underset{\sim}{J}$$

In Coulomb guage

$$\nabla^2 \underset{\sim}{A} = -\mu_0 \underset{\sim}{J}$$

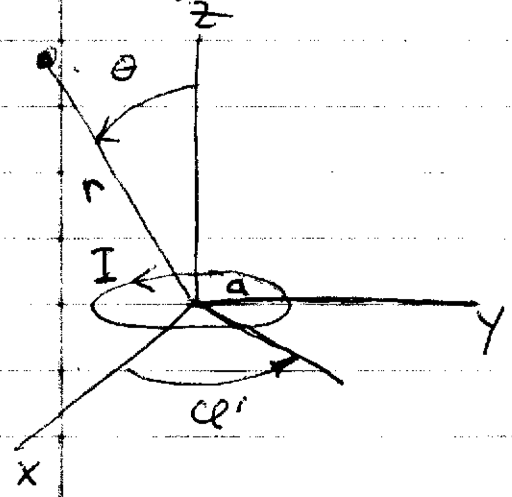
or

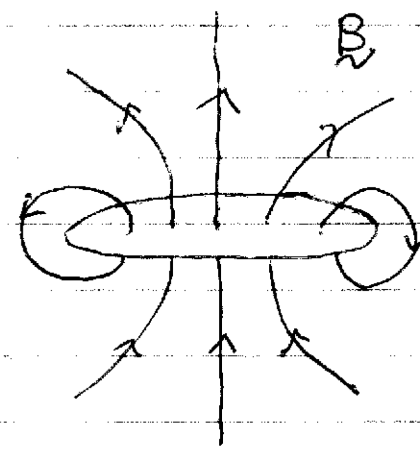
$$\underset{\sim}{A} = \frac{\mu_0}{4\pi} \int d\underset{\sim}{x}' \, \underset{\sim}{J}(\underset{\sim}{x}') \frac{1}{|\underset{\sim}{x} - \underset{\sim}{x}'|}$$

$\underset{\sim}{A}$ in direction of $\underset{\sim}{J}$

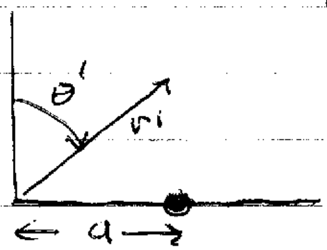
so ψ is a constant and $\nabla \psi = 0$.

Magnetic field from a current loop





$$J_{\varphi'} = I\,\frac{\delta(\theta'-\frac{\pi}{2})\,\delta(r'-a)}{a} \quad \cancel{\neq \frac{I}{a}\,\frac{\delta(\theta'-\frac{\pi}{2})}{\sin\theta'}\,\sin\theta'}$$



$$\int_{a-\epsilon}^{a+\epsilon} dr'\, r' d\theta'\, J_{\varphi'} = I$$

$$\underset{\sim}{J} = -\hat{x}\, J_{\varphi'} \sin\varphi' + \hat{y}\, J_{\varphi'} \cos\varphi'$$

Note $\quad \delta(\theta'-\frac{\pi}{2}) = \frac{\delta(\theta'-\frac{\pi}{2})}{\sin\theta'} \sin\theta' = \delta(\cos\theta')\, \sin\theta'$

Recall: $\quad \int_{-\infty}^{\infty} dx\, \delta[f(x)] = \int_{-\infty}^{\infty} \frac{dx}{df}\, df\, \delta(f)$

$$= \int_{-\infty}^{\infty} df\, \frac{1}{\left|\frac{df}{dx}\right|}\, \delta(f)$$

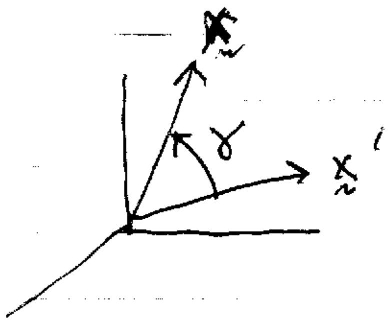
$$J_{\varphi'} = I\, \frac{1}{a}\, \delta(\cos\theta')\, \sin\theta'\, \delta(r'-a)$$

Have symmetry in φ so choose $\varphi = 0$

$\Rightarrow$ only A_y survives

$\Rightarrow$ yields A_φ

$$A_y(r,\theta) = \frac{\mu_0}{4\pi} I \int d(\cos\theta')\, r'^2 dr'\, d\varphi' \frac{1}{|\underset{\sim}{x}-\underset{\sim}{x}'|} \otimes \underbrace{\frac{\delta(\cos\theta')\sin\theta'}{a}\delta(r'-a)\cos\varphi'}_{J_y}$$



$$|\underset{\sim}{x}-\underset{\sim}{x}'| = \left(r^2 + r'^2 - 2rr'\cos\gamma\right)^{1/2}$$

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi')$$

$\Rightarrow$ from electroatic notes

Since $\varphi = 0$, $\theta' = \pi/2$

$$\cos\gamma = \sin\theta\cos\varphi'$$

$$A_y(r,\theta) = \frac{\mu_0}{4\pi} I\, a \int_0^{2\pi} d\varphi' \frac{\cos\varphi'}{\left(r^2+a^2-2ra\sin\theta\cos\varphi'\right)^{1/2}}$$

For $r \gg a$,

$$\left(r^2+a^2-2ra\sin\theta\cos\varphi'\right)^{\frac{1}{2}}$$

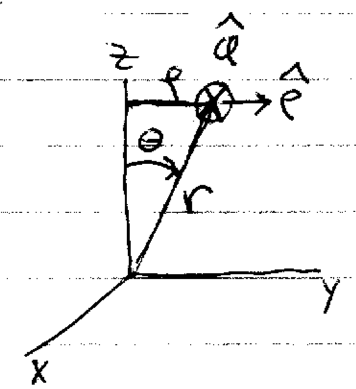
$$\cong r\left(1-\frac{1}{2}\,2\,\frac{a}{r}\sin\theta\cos\varphi'\right)$$

$$A_y = \frac{\mu_0}{4\pi} I \frac{a^2}{r^2} \sin\theta \underbrace{\int_0^{2\pi} d\varphi' \cos^2\varphi'}_{\pi}$$

$$= \frac{\mu_0}{4\pi} \frac{(\pi a^2 I)}{r^2} \sin\theta$$

$$m \equiv \pi a^2 I = \text{magnetic moment}$$

$$\Rightarrow A_\varphi = \frac{\mu_0}{4\pi} \frac{m}{r^2} \sin\theta$$



$$\underset{\sim}{A} = A_\varphi \hat{\varphi}$$

$$\underset{\sim}{B} = \nabla \times (\hat{\varphi} \cdot A_\varphi)$$

$$\nabla\varphi = \frac{1}{\rho}\hat{\varphi} = \frac{1}{r\sin\theta}\hat{\varphi}$$

$$\underset{\sim}{B} = \nabla \times (r\sin\theta A_\varphi \nabla\varphi) = \nabla(r\sin\theta A_\varphi) \times \nabla\varphi$$

since $\nabla \times (\nabla\varphi) = 0$

$$B_r = \left[\nabla_\theta \left(r\sin\theta\, A_\varphi\right)\right] \frac{1}{r\sin\theta}$$

$$\nabla_\theta = \frac{1}{r}\frac{\partial}{\partial\theta}$$

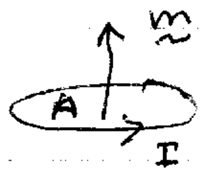
$$B_r = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\sin\theta A_\varphi = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\frac{\mu_0}{4\pi}\frac{m}{r^2}\sin^2\theta$$

$$= \frac{\mu_0}{2\pi}\frac{m\cos\theta}{r^3}$$

$$B_\theta = -\frac{1}{r\cancel{\sin\theta}}\frac{\partial}{\partial r}\left(r\cancel{\sin\theta} A_\varphi\right)$$

$$= -\frac{1}{r}\frac{\partial}{\partial r} r A_\varphi = \frac{\mu_0}{4\pi} m \frac{\sin\theta}{r^3}$$

m = magnetic moment = $I \otimes$ area of loop



Magnetic field from a current loop
$\Rightarrow$ spherical harmonics

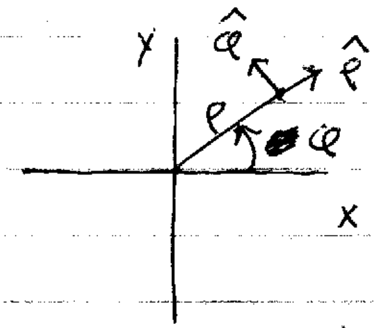
$\nabla^2 \underset{\sim}{A} = -\mu_0 \underset{\sim}{J} \quad \Rightarrow$ Know $\underset{\sim}{A} = A_\varphi \hat{\varphi}$

Take dot product of eqn with $\hat{\varphi}$

$$\hat{\varphi} \cdot \nabla^2 (A_\varphi \hat{\varphi}) = -\mu_0 J_\varphi$$

$$\hat{\varphi} \cdot \nabla^2 (A_\varphi \hat{\varphi}) = \hat{\varphi} \cdot \hat{\varphi} \, \nabla^2 A_\varphi + A_\varphi \, \hat{\varphi} \cdot \nabla^2 \hat{\varphi}$$

$$\nabla^2 \hat{\varphi} = \frac{1}{r^2 \sin^2\theta} \frac{\partial^2}{\partial \varphi^2} \hat{\varphi}$$



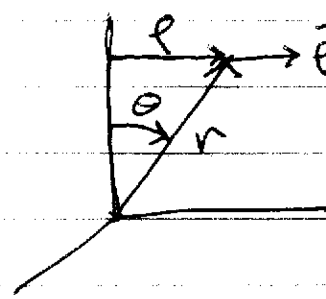
$$\frac{\partial}{\partial \varphi} \hat{\varphi} = -\hat{\rho}$$

$$\frac{\partial}{\partial \varphi} \hat{\rho} = \hat{\varphi}$$

$\rho = r \sin\theta$

$$\nabla^2 \hat{\varphi} = -\frac{1}{r^2 \sin^2\theta} \hat{\varphi}$$

$$\hat{\varphi} \cdot \nabla^2 \hat{\varphi} = -\frac{1}{r^2 \sin^2\theta}$$



$$\nabla^2 A_\varphi \underbrace{- \frac{1}{r^2 \sin^2\theta} A_\varphi}_{} = -\mu_0 J_\varphi$$

This term did not appear in Poisson's eqn. for azimuthally symmetric systems.

$\Rightarrow \underset{\sim}{A} = A_\varphi \hat{\varphi}$ is a vector rather than a scalar.

$$\nabla^2 A_\varphi = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial A_\varphi}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} A_\varphi$$

$$\Rightarrow \quad \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} A_\varphi + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} A_\varphi$$

$$- \frac{1}{r^2 \sin^2\theta} A_\varphi = -\mu_0 J_\varphi$$

with

$$J_\varphi = I \frac{\sin\theta}{a}\delta(\cos\theta)\,\delta(r-a)$$

In Poisson's equ for the scalar potential, such a term would only arise if

$$\frac{\partial^2}{\partial\varphi^2} = -m^2 = -1$$

The vector $A_\varphi \hat{\varphi}$ acts as if $m = 1$

$\Rightarrow$ changes basis function for equ

$\Rightarrow$ $Y_{\ell 1}(\theta, 0)$ are basis functions

$$Y_{\ell 1}(\theta, 0) = \frac{2\ell+1}{4\pi}\frac{(\ell-1)!}{(\ell+1)!} P_{\ell 1}(\cos\theta)$$

$$A_\varphi = \sum_\ell Y_{\ell 1}(\theta, 0)\, g_\ell(r)$$

$Y_{\ell 1}(\theta, 0)$ are a complete set over
$\theta = (0, \pi)$

$\Rightarrow$ orthonormal
$\Rightarrow$ carry out usual procedure to obtain eqn for $g_\ell(r)$
$\Rightarrow$ find jump conditions across $r = a$.

$$A_\varphi = \frac{\mu_0 I a}{2} \sum_{\ell \, odd} \frac{r_<^\ell}{r_>^{\ell+1}} \frac{1}{\ell(\ell+1)} P_{\ell 1}(0)$$

$$\otimes \; P_{\ell 1}(\cos\theta)$$

For large r $\Rightarrow$ $\ell = 1$ dominates

$$P_{11}(\cos\theta) = -\sin\theta, \quad P_{11}(0) = -1$$

$$A_\varphi = \frac{\mu_0}{4\pi} m \frac{\sin\theta}{r^2} \quad \text{as before.}$$