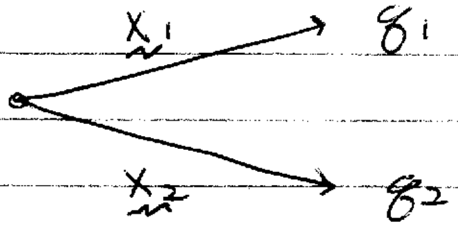
Electrostatics (Jackson Ch 1)

Coulomb's Law (SI units)



$$\underset{\sim}{F}_1 = \frac{q_1 q_2}{4\pi\epsilon_0} \frac{\underset{\sim}{x}_1 - \underset{\sim}{x}_2}{|\underset{\sim}{x}_1 - \underset{\sim}{x}_2|^3}$$

$\Rightarrow$ like charges repell

$$\epsilon_0 = 8.85 \times 10^{-12} \frac{C^2}{Nm^2} = \text{permittivity}$$

$$k \equiv \frac{1}{4\pi\epsilon_0} \simeq 9 \times 10^{9} \frac{Nm^2}{C^2}$$

SI units	CGS units
q_1, q_2 Coulombs	$1\,C = 3 \times 10^9$ stat C
$\underset{\sim}{x}_1, \underset{\sim}{x}_2$ meters	$1\,m = 100\,cm$
$\underset{\sim}{F}$, Newtons	$1\,N = 10^5$ dynes
energy, Joules	$1\,J = 10^7$ ergs

Electric Field $\underset{\sim}{F} = q\,\underset{\sim}{E}$

$\Rightarrow \underset{\sim}{E}$ points from positive to negative charge

$$\underset{\sim}{E}(\underset{\sim}{x}) = \frac{1}{4\pi\epsilon_0} \sum_j q_j \frac{(\underset{\sim}{x} - \underset{\sim}{x}_j')}{|\underset{\sim}{x} - \underset{\sim}{x}_j'|^3} \sim \frac{\text{volts}}{m}$$

For continuous distributions

$$\underset{\sim}{E}(\underset{\sim}{x}) = \frac{1}{4\pi\epsilon_0} \int d\underset{\sim}{x}' \, \rho(\underset{\sim}{x}') \frac{\underset{\sim}{x} - \underset{\sim}{x}'}{|\underset{\sim}{x} - \underset{\sim}{x}'|^3}$$

$$d\underset{\sim}{x}' = dx'dy'dz'$$

$$\rho(\underset{\sim}{x}) = \text{charge density} = \frac{\text{charge}}{\text{vol}}$$

For discrete charges

$$\rho(\underset{\sim}{x}) = \sum_j q_j \, \delta(\underset{\sim}{x} - \underset{\sim}{x}_j')$$

$$\delta(\underset{\sim}{x} - \underset{\sim}{x}_j') = \delta(x - x_j')\,\delta(y - y_j')\,\delta(z - z_j')$$

$$\int_V d\underset{\sim}{x}' \, \delta(\underset{\sim}{x} - \underset{\sim}{x}_j') = \begin{cases} 1 & \text{if } \underset{\sim}{x}_j' \in V \\ 0 & \text{if } \underset{\sim}{x}_j \notin V \end{cases}$$

Scalar Potential

$$\nabla \frac{1}{|\underset{\sim}{x} - \underset{\sim}{x}'|} = \nabla \frac{1}{[(\underset{\sim}{x} - \underset{\sim}{x}')\cdot(\underset{\sim}{x} - \underset{\sim}{x}')]^{1/2}}$$

$$= -\frac{1}{2} \frac{1}{|\underset{\sim}{x} - \underset{\sim}{x}'|^3} \nabla [(\underset{\sim}{x} - \underset{\sim}{x}')\cdot(\underset{\sim}{x} - \underset{\sim}{x}')]$$

$$= -\frac{1}{2}\frac{1}{|\underset{\sim}{x}-\underset{\sim}{x}'|^3}\left(2\underset{\sim}{x}-2\underset{\sim}{x}'\right)$$

$$= -\frac{\underset{\sim}{x}-\underset{\sim}{x}'}{|\underset{\sim}{x}-\underset{\sim}{x}'|^3}$$

$$\underset{\sim}{E} = -\nabla\int d\underset{\sim}{x}'\,\frac{1}{4\pi\epsilon_0}\,\rho(\underset{\sim}{x}')\,\frac{1}{|\underset{\sim}{x}-\underset{\sim}{x}'|}$$

Let

$$\varphi \equiv \frac{1}{4\pi\epsilon_0}\int d\underset{\sim}{x}'\,\frac{\rho(\underset{\sim}{x}')}{|\underset{\sim}{x}-\underset{\sim}{x}'|}$$

$$\underset{\sim}{E} = -\nabla\varphi(\underset{\sim}{x})$$

$\Rightarrow$ $\underset{\sim}{E}$ points down the potential

$\varphi \sim$ volts

$\varphi(\underset{\sim}{x})$ is easier to evaluate than $\underset{\sim}{E}$ since it is a scalar.

Energy of a charge in a potential

Move a charge from $\underset{\sim}{x}_1$ to $\underset{\sim}{x}_2$ quasistatically so no change in KE.

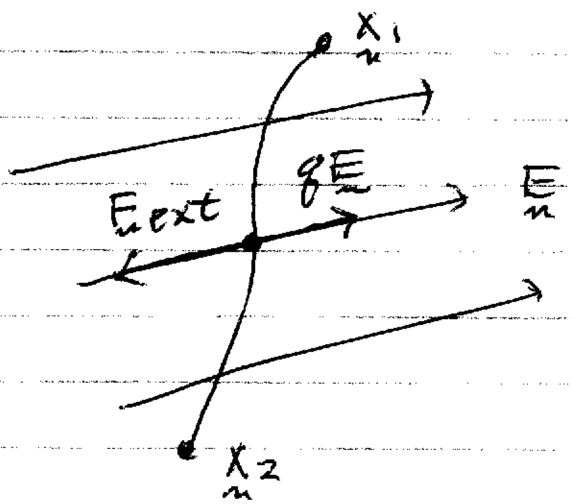
$$m\dot{\underset{\sim}{v}} = q\underset{\sim}{E} + \underset{\sim}{F}_{ext} \simeq 0$$

$\underset{\sim}{F}ext$ is the force needed to move the particle through the change in potential. Work done by external force is

$$W_{ext} = \int_{\underset{\sim}{x}_1}^{\underset{\sim}{x}_2} d\underset{\sim}{x} \cdot \underset{\sim}{F}_{ext} = q \int_{\underset{\sim}{x}_1}^{\underset{\sim}{x}_2} d\underset{\sim}{x} \cdot \nabla \Phi$$

$$= q \left[\Phi(\underset{\sim}{x}_2) - \Phi(\underset{\sim}{x}_1) \right]$$

$\Rightarrow q\,\Phi(\underset{\sim}{x})$ = potential energy of a charge q at $\underset{\sim}{x}$.



Note that

$$\underset{\sim}{E} = -\nabla\Phi$$

implies that

$$\nabla \times \underset{\sim}{E} = 0$$

$$\Rightarrow \oint \underset{\sim}{E} \cdot d\underset{\sim}{x} = 0$$

Poisson's Eqn

First consider

$$\nabla \cdot \underset{\sim}{E} = -\int d\underset{\sim}{x}' \frac{1}{4\pi\epsilon_0} \rho(\underset{\sim}{x}') \nabla^2 \frac{1}{|\underset{\sim}{x} - \underset{\sim}{x}'|}$$

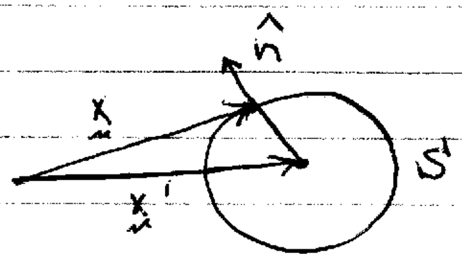
$$\nabla\cdot\nabla\frac{1}{|\underset{\sim}{x}-\underset{\sim}{x}'|} = -\nabla\cdot\frac{\underset{\sim}{x}-\underset{\sim}{x}'}{|\underset{\sim}{x}-\underset{\sim}{x}'|^3}$$

$$= -\frac{1}{|\underset{\sim}{x}-\underset{\sim}{x}'|^3}\underbrace{\nabla\cdot\underset{\sim}{x}}_{3} + \frac{3}{2}\frac{(\underset{\sim}{x}-\underset{\sim}{x}')}{|\underset{\sim}{x}-\underset{\sim}{x}'|^5}\underbrace{\nabla|\underset{\sim}{x}-\underset{\sim}{x}'|^2}_{2(\underset{\sim}{x}-\underset{\sim}{x}')}$$

$$= -\frac{3}{|\underset{\sim}{x}-\underset{\sim}{x}'|^3} + \frac{3}{|\underset{\sim}{x}-\underset{\sim}{x}'|^3} = 0$$

However,

$$I \equiv \int d\underset{\sim}{x}\,\nabla\cdot\nabla\frac{1}{|\underset{\sim}{x}-\underset{\sim}{x}'|} = \underbrace{\int_{S} d\underset{\sim}{s}\,\hat{n}\cdot\nabla\frac{1}{|\underset{\sim}{x}-\underset{\sim}{x}'|}}_{\text{divergence theorem}}$$



S is a spherical shell centered on $\underset{\sim}{x}'$

$\hat{n}$ = radial unit vector

$\hat{n}\cdot\hat{n} = 1$

$$I = -\int_{S} ds\,\frac{\hat{n}\cdot(\underset{\sim}{x}-\underset{\sim}{x}')}{|\underset{\sim}{x}-\underset{\sim}{x}'|^3} = -\frac{4\pi|\underset{\sim}{x}-\underset{\sim}{x}'|^2}{|\underset{\sim}{x}-\underset{\sim}{x}'|^2}$$

$$= -4\pi$$

$$\Rightarrow \nabla^2 \frac{1}{|\underset{\sim}{x}-\underset{\sim}{x}'|} = -4\pi\,\delta(\underset{\sim}{x}-\underset{\sim}{x}')$$

$$\nabla\cdot\underset{\sim}{E} = \int d\underset{\sim}{x}'\,\frac{\rho(\underset{\sim}{x}')}{4\pi\epsilon_0}\,4\pi\,\delta(\underset{\sim}{x}-\underset{\sim}{x}')$$

$$\nabla\cdot\underset{\sim}{E} = \frac{1}{\epsilon_0}\,\rho(\underset{\sim}{x})$$ Differential form of Gauss' Law

Since $\underset{\sim}{E} = -\nabla\varphi$

$$\nabla^2\varphi = -\frac{1}{\epsilon_0}\,\rho(\underset{\sim}{x})$$ Poisson's Eqn

If $\rho(\underset{\sim}{x}) = 0$,

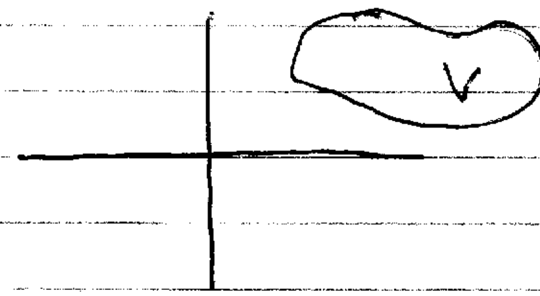
$$\nabla^2\varphi = 0$$ Laplace's Eqn

Gauss' Law

Consider

$$\nabla \cdot \underset{\sim}{E} = \frac{1}{\epsilon_0} \rho$$

Integrate over a closed volume of arbitrary shape



$$\int_V d\underset{\sim}{x}\, \nabla \cdot \underset{\sim}{E} = \int_V d\underset{\sim}{x}\, \frac{\rho(\underset{\sim}{x})}{\epsilon_0}$$

From the divergence theorem

$$\int_V d\underset{\sim}{x}\, \nabla \cdot \underset{\sim}{E} = \int_S dS\, \underset{\sim}{E} \cdot \hat{n}$$

$\hat{n}$ is the outward normal to the surface element dS

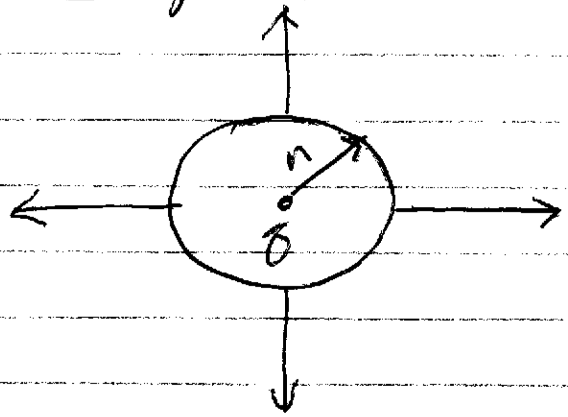
$$\int_S dS\, \underset{\sim}{E} \cdot \hat{n} = \frac{1}{\epsilon_0} \int_V d\underset{\sim}{x}\, \rho(\underset{\sim}{x})$$

$$= \frac{1}{\epsilon_0} \times \text{total charge enclosed}$$

The total electric flux out of a ~~surface~~ volume depends only on the total charge enclosed.

The integral form of Gauss' law is useful in cases where the system has a high degree of symmetry.

Example Back to a point charge

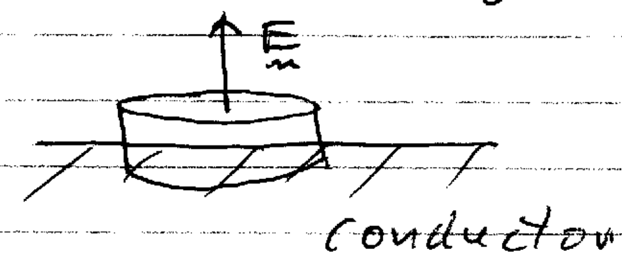


$$E\, 4\pi r^2 = \frac{q}{\epsilon_0}$$

$$E = \frac{q}{4\pi\epsilon_0}\frac{1}{r^2}$$

Example Conductor

A conductor is a material in which charge can move in response to $\underset{\sim}{E}$. In a steady state, $\underset{\sim}{E} = 0$ inside a conductor



Consider a pillbox of area ΔS and infinitesimal height.

$$\int_{\Delta S} \underset{\sim}{E}\cdot\hat{n}\, dS = \underset{\sim}{E}\cdot\hat{n}\,\Delta S = \frac{1}{\epsilon_0}\int d\underset{\sim}{x}'\, \rho(\underset{\sim}{x}')$$

$\rho(\underset{\sim}{x}') = 0$ inside a conductor since

$$\underset{\sim}{E} = 0 \Rightarrow \nabla\cdot\underset{\sim}{E} = 0 \Rightarrow \rho = 0$$

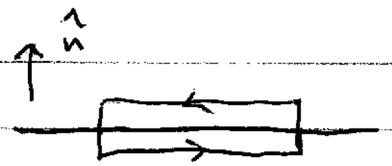
$\Rightarrow$ can only have a surface charge $\sigma(\underset{\sim}{x}) = \frac{\text{charge}}{\text{area}}$

$$\underset{\sim}{E}\cdot\hat{n}\,\Delta S = \frac{1}{\epsilon_0}\sigma\,\Delta S$$

$$\underset{\sim}{E} \cdot \hat{n} = \frac{1}{\epsilon_0} \sigma$$

What about the tangential of $\underset{\sim}{E}$ on a conductor?

$$\nabla \times \underset{\sim}{E} = 0 \implies \underbrace{\oint \underset{\sim}{E} \cdot d\underset{\sim}{\ell} = 0}_{\text{Stoke's Theorem}}$$



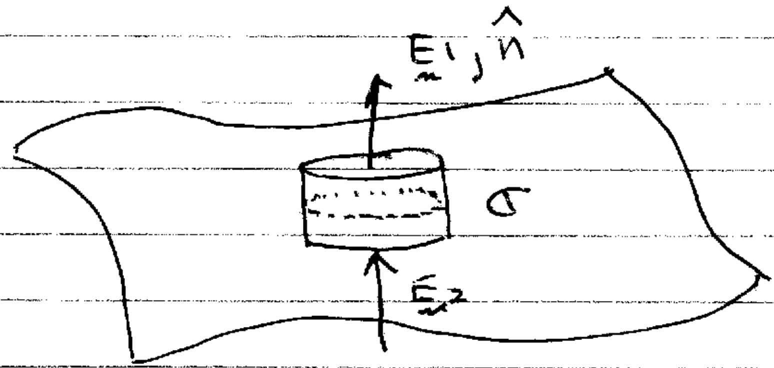
$\Rightarrow$ $\underset{\sim}{E} \times \hat{n}$ continuous across surface

but $\underset{\sim}{E} = 0$ inside

$\Rightarrow$ $\underset{\sim}{E} \times \hat{n} = 0$ just outside the conductor

$\Rightarrow$ $\underset{\sim}{E}$ is normal to the surface of a conductor.

<u>Example</u> Surface Charge



From flat sheet of charge σ

$\underset{\sim}{E}_1 = -\underset{\sim}{E}_2$

$E_n = \frac{\sigma}{2\epsilon_0}$

$$\int_{\Delta S} dS\, \underset{\sim}{E} \cdot \hat{n} = (\underset{\sim}{E}_1 - \underset{\sim}{E}_2) \cdot \hat{n}\, \Delta S = \frac{1}{\epsilon_0} \sigma \Delta S$$

$$(\underset{\sim}{E}_1 - \underset{\sim}{E}_2) \cdot \hat{n} = \frac{\sigma}{\epsilon_0}$$ Discontinuity in $\underset{\sim}{E}$

Tangential component? $\Rightarrow$ continuous

Uniqueness of solutions of Poisson's Eqn

If have an infinite system with no boundaries can write

$$\Phi = \frac{1}{4\pi\epsilon_0} \int d\underset{\sim}{x}' \, \frac{\rho(\underset{\sim}{x}')}{|\underset{\sim}{x}-\underset{\sim}{x}'|}$$

to calculate Φ, $\underset{\sim}{E}$. If have boundaries where specify Φ or $\underset{\sim}{E}$, other approaches are better. Can solve

$$\nabla^2 \Phi = -\frac{1}{\epsilon_0} \rho(\underset{\sim}{x})$$

and impose BCs.

Suppose have a volume V with a charge distribution ρ and want to specify BCs on Φ or its derivatives. Under what conditions is the solution unique? Overdetermined?

Consider two solutions

$$\nabla^2 \Phi_1 = -\frac{1}{\epsilon_0} \rho(\underset{\sim}{x})$$

$$\nabla^2 \Phi_2 = -\frac{1}{\epsilon_0} \rho(\underset{\sim}{x})$$

Suppose $\Phi_1 \neq \Phi_2$ and let $U \equiv \Phi_2 - \Phi_1$

Then $\nabla^2 u = 0$ and

$$\int_V d\underset{\sim}{x}\, u \nabla^2 u = 0$$

but

$$u \nabla^2 u = \nabla \cdot (u \nabla u) - |\nabla u|^2 = 0$$

$$\int_V d\underset{\sim}{x}\, |\nabla u|^2 = \int d\underset{\sim}{x}\, \nabla \cdot (u \nabla u) = \int_S ds\, \hat{n} \cdot (u \nabla u)$$

$$= \int ds\, u \frac{\partial u}{\partial n}$$

(1) Suppose specify Φ on boundary

$\Rightarrow \Phi_1 = \Phi_2$ on S

$\Rightarrow u = 0$ on S

$\Rightarrow \int_V d\underset{\sim}{x}\, |\nabla u|^2 = 0$

$\Rightarrow u =$ const. but $u = 0$ on S

so $u = 0$ everywhere and

$\Phi_1(\underset{\sim}{x}) = \Phi_2(\underset{\sim}{x})$

$\Rightarrow$ solutions of Poisson's eqn for Φ specified on boundary are unique

$\Rightarrow$ Dirichlet BC's

2. Specify $\mathbf{E}\cdot\hat{n}$ on the boundary

$$\Rightarrow \quad \frac{\partial \Phi_1}{\partial n} = \frac{\partial \Phi_2}{\partial n} \quad \text{on } S$$

Again

$$\int_V d\mathbf{x}\, |\nabla u|^2 = 0 \quad \text{or}$$

$$u = \text{const}$$

$$\Phi_2 - \Phi_1 = \text{const.}$$

If specify $\mathbf{E}\cdot\hat{n}$ on S and solve

$$\nabla^2 \Phi = -\frac{1}{\epsilon_0}\rho$$

the solution is unique within an additive constant

$\Rightarrow$ Neumann BCs

$\Rightarrow$ can have mixed N conditions ~~over~~ and D conditions as long as they don't overlap.

$\Rightarrow$ overdetermined if both specified on a surface

<u>Example</u> Conducting shell with external charge

$\rho(\underset{\sim}{x})$

Let $\rho(\underset{\sim}{x}) = 0$ interior to shell.
What is $\underset{\sim}{E}$ inside of shell?

Since $\underset{\sim}{E} \times \hat{n} = 0$ at surface of shell,
$\Phi = $ const. on shell
$\Rightarrow$ conductors are equipotentials

Inside the shell have
$\nabla^2 \Phi = 0$ ~~wh~~ with $\Phi = \Phi_0$ on the boundary

$\Phi = \Phi_0$ satisfies $\nabla^2 \Phi = 0$ inside and the BCs. Solution unique so

$$\underset{\sim}{E} = -\nabla \Phi = 0 \quad \text{inside}$$

$\Rightarrow$ A closed conducting shell shields external electric fields.

$\Rightarrow$ Does a shell shield the outside from charge inside?