The invariance of Maxwell's eqns under the Lorentz transformation

We would like to write Maxwell's equations in a form that is independent of reference frame. To do this we need to discuss the transformation properties of scalars, vectors and tensors since in the end we will show that $\underline{E}$ and $\underline{B}$ are components of a second rank tensor.

Lorentz Group

In 3D rotations can be represented by the group of transformations that leave the norm of the vector $\underline{x}$ invariant. In special relativity the class of all transformations that leave

$$s^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2$$

invariant are members of the Lorentz group. These include both rotations and Lorentz transformations.

From the first postulate we know that equations describing the laws of nature must be covariant, that is invariant under transformations of the Lorentz group. (LG)

We define scalars, vectors and tensors according to their transformation properties under the LG.

A scalar (tensor of rank zero) is a quantity whose value is not changed by the transformation

$$\Rightarrow s^2 = s \cdot s \text{ is a Lorentz scalar}$$

A vector (tensor of rank one) falls into two classes. First consider contravariant vectors e.g. $(ct, \underset{\sim}{x}) = x^{\mu}$

We can represent the transformation in matrix form

$$x^{\mu'} = A^{\mu}{}_{\nu} x^{\nu} = \frac{\partial x^{\mu'}}{\partial x^{\nu}} x^{\nu}$$

For a LT in the x direction

$$\begin{pmatrix} x^{0'} \\ x^{1'} \\ x^{2'} \\ x^{3'} \end{pmatrix} = \underbrace{\begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{A^{\mu}{}_{\nu}} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

Similar matrices describe rotations and LTs in y, z.

We also have covariant vectors. How do they transform? We first define a matrix that transforms a contravariant vector to a covariant vector

$$x_\mu = g_{\mu\nu} x^\nu$$

$$g_{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

$$\Rightarrow \quad x_\mu = (ct, -\underset{\sim}{x})$$

We can use g to obtain an equation describing the transformation of x_μ.

The matrix equation for contravariant vectors is

$$x' = A x$$

with A given previously. Operate on both sides with g,

$$g x' = g A g g x = (gAg)(gx)$$

where g^2 is the identity matrix.

$$A_\mu{}^\nu = g_{\mu\alpha} A^\alpha{}_\beta \, g^{\beta\nu}$$

where $A_\mu{}^\nu$ is the transformation matrix for covariant vectors

$$x'_\mu = A_\mu{}^\nu x_\nu$$

So

$$A_\mu{}^\nu = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \begin{pmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & 0 \\ & 0 & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ -\beta\gamma & -\gamma & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The matrix g is also the norm or metric

$$ds^2 = dx^\alpha dx_\alpha$$
$$= dx^\alpha g_{\alpha\beta} dx^\beta$$

Tensors

The transformation of tensors follows from the transformation properties of vectors

$$T'^{\mu\alpha} = x'^\mu x'^\alpha = A^\mu{}_\nu x^\nu A^\alpha{}_\beta x^\beta$$
$$= A^\mu{}_\nu x^\nu x^\beta A_\beta{}^\alpha$$
$$= A T A^T$$

Derivatives

$$\frac{\partial}{\partial x^{0\prime}} = \frac{\partial x^0}{\partial x^{0\prime}} \frac{\partial}{\partial x^0} + \frac{\partial x^1}{\partial x^{0\prime}} \frac{\partial}{\partial x^1} + \cdots$$

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^{0\prime} \\ x^{1\prime} \\ x^{2\prime} \\ x^{3\prime} \end{pmatrix}$$

$$\frac{\partial}{\partial x^{0\prime}} = \gamma \frac{\partial}{\partial x^0} + \beta\gamma \frac{\partial}{\partial x^1}$$

$$\frac{\partial}{\partial x'^{\nu}} = \ldots$$

$\Rightarrow$ derivative with respect to a contravariant vector transforms like a covariant vector

$$\left(\frac{\partial}{\partial ct}, \nabla\right) = \partial_\mu$$

Divergence

$$\partial_\alpha A^\alpha = \frac{\partial}{\partial x^0} A^0 + \nabla \cdot \underset{\sim}{A}$$

4-D Laplacian

$$\Box = \partial_\alpha \partial^\alpha = \partial_\alpha g^{\alpha\beta} \partial_\beta$$

$$= \frac{\partial^2}{\partial x^{0 2}} - \nabla^2$$

Covariance of Maxwell's eqns

We want to show that Maxwell's eqns can be written in a form that is independent of reference frame. We begin with charge continuity

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \underset{\sim}{J} = 0$$

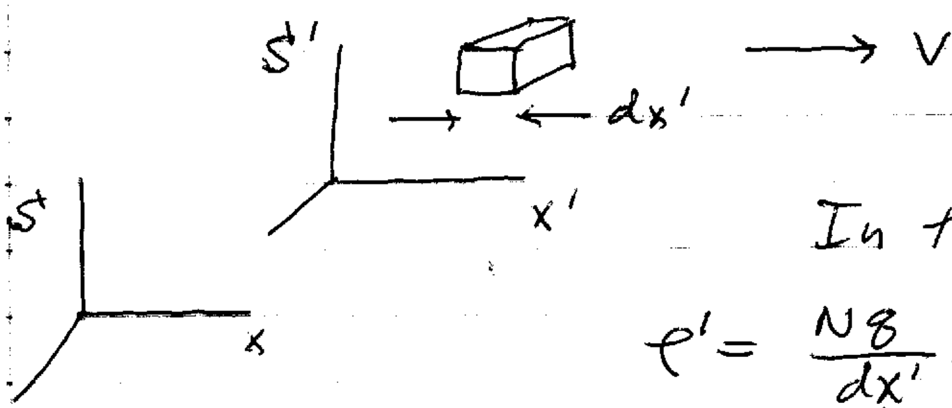
This can be written as

$$\frac{\partial}{\partial x^0} c\rho + \nabla \cdot \underset{\sim}{J} = 0$$

If $(c\rho, \underset{\sim}{J})$ is a 4-vector then the continuity equation can be written as

$$\partial_\alpha J^\alpha = 0,$$

which is covariant. We want to show that $c\rho$ and $\underset{\sim}{J}$ form a 4-vector. Consider a frame S' in which we have a volume element $d\underset{\sim}{x}'$ containing N charges at rest. Observations reveal that charge is a Lorentz scalar



In the S' frame

$$\rho' = \frac{Nq}{d\underset{\sim}{x}'} = \frac{Nq}{dx'dy'dz'}$$

In the S' frame

$$\underset{\sim}{J}' = 0$$

$\Rightarrow$ the charges are stationary

with $dx = \frac{dx'}{\gamma}$

$dy = dy'$

$dz = dz'$

In the frame S we have

$$\rho = \frac{Nq}{d\underset{\sim}{x}} = \frac{Nq\,\gamma}{dx'dy'dz'} = \gamma\rho'$$

The current density J_x is the charge crossing a unit area per unit time. The N charges cross the area $dydz$ in a time

$$dt = \frac{dx}{v}$$

$$J_x = \frac{Nq}{dydz}\frac{1}{dt} = \frac{Nqv}{dxdydz} = \rho v$$

$$J_x = \gamma\rho' v$$

To show that $(c\rho, \underset{\sim}{J})$ is a 4-vector we need to show that its length is invariant under LT.

$$c^2\rho^2 - J_x^2 = c^2\gamma^2\rho'^2 - \gamma^2\rho'^2 v^2$$

$$= c^2\gamma^2\rho'^2(1-\beta^2)$$

$$= c^2\rho'^2$$

Thus, $(c\rho, \underset{\sim}{J})\cdot(c\rho\ \underset{\sim}{J})$ is a scalar and

$(c\rho, \underset{\sim}{J})$ forma 4-vector

In the Lorentz guage the wave equations for $\underset{\sim}{A}$ and φ are

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\underset{\sim}{A} - \nabla^2 \underset{\sim}{A} = \mu_0 \underset{\sim}{J}$$

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\varphi - \nabla^2 \varphi = \frac{\rho}{\epsilon_0} = \mu_0 c^2 \rho$$

$$\frac{\partial}{\partial ct}\left(\frac{\varphi}{c}\right) + \nabla\cdot\underset{\sim}{A} = 0$$

Thus, $\frac{\varphi}{c}$ and $\underset{\sim}{A}$ must form a 4-vector and the wave equs and the Lorenz condition can be written as

$$\Box A^\alpha = \mu_0 J^\alpha \qquad \text{Wave equs}$$

$$\partial_\alpha A^\alpha = 0 \qquad \text{Lorenz cond.}$$

<u>Transformation of Electric and Magnetic Fields</u>

$\underset{\sim}{E}$ and $\underset{\sim}{B}$ are obtained through gradients of $\underset{\sim}{A}$ and φ,

$$\underset{\sim}{B} = \nabla\times\underset{\sim}{A}$$

$$\underset{\sim}{E} = -\frac{\partial}{\partial t}\underset{\sim}{A} - \nabla\varphi$$

Since $A, \varphi/c$ form a 4-vector and $\underset{\sim}{E}$ and $\underset{\sim}{B}$ are gradients of this 4-vector, $\underset{\sim}{E}$ and $\underset{\sim}{B}$ must not transform as a 4-vector

To see how $\underset{\sim}{E}$, $\underset{\sim}{B}$ transform we show ~~a few~~ examples of ~~the~~ components of $\underset{\sim}{E}$ and $\underset{\sim}{B}$. For example,

$$E_x = -c\left[\frac{\partial}{\partial ct} A_x + \frac{\partial}{\partial x} \frac{\varphi}{c} \right]$$

$$= -c\left[\partial^0 A^1 - \partial^1 A^0 \right]$$

where $\partial^\alpha = \left(\frac{\partial}{\partial x_0}, -\nabla \right)$. Similarly for B_x,

$$B_x = \frac{\partial}{\partial y} A_z - \frac{\partial}{\partial z} A_y$$

$$= -\left(\partial^2 A^3 - \partial^3 A^2 \right)$$

Thus, $\underset{\sim}{E}$ and $\underset{\sim}{B}$ are elements of a second rank tensor

$\Longrightarrow$ field strength tensor

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$$

$$F^{\alpha\beta} = \left(\begin{array}{c|ccc} 0 & -\frac{E_x}{c} & -\frac{E_y}{c} & -\frac{E_z}{c} \\ \hline \frac{E_x}{c} & 0 & -B_z & B_y \\ \frac{E_y}{c} & B_z & 0 & -B_x \\ \frac{E_z}{c} & -B_y & B_x & 0 \end{array}\right)$$

$$= \left(\begin{array}{c|c} 0 & -\underset{\sim}{E}/c \\ \hline \frac{\underset{\sim}{E}}{c} & -\epsilon_{ijk} B_k \end{array}\right)$$

where

$$\epsilon_{123} = 1 \,,\ \epsilon_{213} = -1 \,,\ \epsilon_{312} = 1$$

$$\epsilon_{ijk} = \begin{cases} 0 & \text{if } i=j \text{ or } j=k \text{ or } i=k \\ +1 & \text{even permutation} \\ -1 & \text{odd permutation} \end{cases}$$

$$-\epsilon_{ijk} B_k = \begin{pmatrix} \cancelto{0}{-\epsilon_{11k} B_k} & -\epsilon_{123} B_3 & -\epsilon_{132} B_2 \\ -\epsilon_{213} B_3 & \cancelto{0}{-\epsilon_{22k} B_k} & -\epsilon_{231} B_1 \\ -\epsilon_{312} B_2 & -\epsilon_{321} B_1 & \cancelto{0}{-\epsilon_{33k} B_k} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -B_3 & B_2 \\ B_3 & 0 & -B_1 \\ -B_2 & B_1 & 0 \end{pmatrix}$$

Recalling $\partial^\alpha = \left(\frac{\partial}{\partial x_0}, -\nabla\right)$, $\partial_\alpha = \left(\frac{\partial}{\partial x_0}, \nabla\right)$

Starting with Maxwell's eqns,

$$\nabla \cdot \underset{\sim}{E} = \frac{\rho}{\epsilon_0} = \mu_0 \frac{\rho}{\mu_0 \epsilon_0} = \mu_0 (c\rho) c$$

$$\nabla \cdot \left(\frac{\underset{\sim}{E}}{c}\right) = \mu_0 c\rho$$

$$\partial_\alpha F^{\alpha 0} = \mu_0 J^0 \Leftarrow (1)$$

$$\left(\nabla \times \underset{\sim}{B} - \frac{1}{c^2} \frac{\partial}{\partial t} \underset{\sim}{E} = \mu_0 \underset{\sim}{J}\right)_z$$

$$\frac{\partial}{\partial x} B_y - \frac{\partial}{\partial y} B_x - \frac{1}{c^2} \frac{\partial}{\partial t} E_z = \mu_0 J_z$$

$$\partial_1 F^{13} + \partial_2 F^{23} + \partial_0 F^{03} = \mu_0 J^3$$

$$\partial_\alpha F^{\alpha 3} = \mu_0 J^3 \Leftarrow (2)$$

Eqns (1) and (2) are

$$\boxed{\partial_\alpha F^{\alpha\beta} = \mu_0 J^\beta}$$

This yields the two Maxwell's eqns with sources in covariant form.

Procedure for is similar,

$$\left.\begin{array}{l} \nabla \cdot \underset{\sim}{B} = 0 \\ \nabla \times \underset{\sim}{E} + \frac{\partial}{\partial t} \underset{\sim}{B} \end{array}\right\} \quad \partial^\alpha F^{\beta\gamma} + \partial^\beta F^{\gamma\alpha} + \partial^\gamma F^{\alpha\beta} = 0$$

where $(\alpha, \beta, \gamma) =$

$$\left.\begin{array}{c} (1, 2, 3) \\ (2, 3, 0) \\ (3, 0, 1) \\ (0, 1, 2) \end{array}\right\} \text{4 eqns}$$

This completes the establishment of Maxwell's eqns in covariant form

After discussing the covariance of the momentum and energy eqns., we will write explicit eqns for the ~~the~~ transformation of $\underset{\sim}{E}$, $\underset{\sim}{B}$ under frame shifts.

Covariance Momentum and energy equations

The ~~for~~ momentum and energy equations are as follows:

$$\frac{d}{dt} \underset{\sim}{p} = q\left(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}\right) = \underset{\sim}{F}$$

$$\frac{d}{dt} \mathcal{E} = \underset{\sim}{F} \cdot \underset{\sim}{v} = q \underset{\sim}{E} \cdot \underset{\sim}{v}$$

where $\mathcal{E}$ is the energy of particle. $d\underline{P}/dt$ is not a 4-vector but $d\underline{P}/d\tau$ with $d\tau = dt/\gamma$ is a component of a 4-vector. Thus, the momentum eqn becomes

$$\frac{d\underline{P}}{d\tau} = q\underline{E}\gamma + q\gamma\,\underline{V}\times\underline{B}$$

We can define a 4-velocity

$$u = \frac{1}{m}P = \frac{1}{m}\left(\frac{\mathcal{E}}{c}, \underline{P}\right)$$

$$= \frac{1}{m}(\gamma mc, \underline{P})$$

so

$$u^\alpha = (\gamma c, \gamma\underline{V})$$

$$u_\alpha = (\gamma c, -\gamma\underline{V})$$

Consider the x component of the momentum eqn,

$$\frac{d}{d\tau}P_x = q\gamma c\frac{E_x}{c} + q\gamma(V_yB_z - V_zB_y)$$

$$\frac{d}{d\tau}P^1 = q\left[F^{10}u_0 + F^{12}u_2 + F^{13}u_3\right]$$

$$= qF^{1\beta}u_\beta$$

The energy eqn can be written as

$$\frac{d\mathcal{E}}{d\tau} = q\, \underset{\sim}{E} \cdot \underset{\sim}{u}$$

$$\frac{d}{d\tau} P^0 = q\, \frac{\underset{\sim}{E}}{c} \cdot \underset{\sim}{u} = q\, F^{0\beta} u_\beta$$

The momentum/energy eqn is then

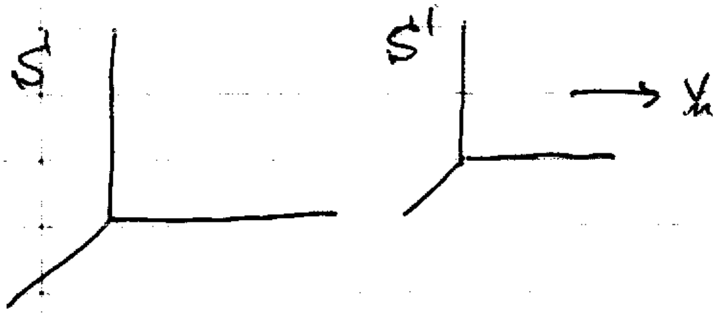
$$\boxed{\frac{d}{d\tau} P^\alpha = q\, F^{\alpha\beta} u_\beta}$$

Transformation of electromagnetic fields

Since $\underset{\sim}{E}$ and $\underset{\sim}{B}$ are the components of 2nd rank tensors, the transformation of $\underset{\sim}{E}, \underset{\sim}{B}$ is given by

$$\underset{\sim}{F}' = A F A^t$$

or $$F'^{\alpha\beta} = A^\alpha{}_\gamma F^{\gamma\epsilon} A_\epsilon{}^\beta$$



The transformations formulas are

$$\frac{1}{c} E_{\parallel}' = \frac{1}{c} E_{\parallel}$$
$$B_{\parallel}' = B_{\parallel}$$

$\parallel$ is along the transformation direction

$$\frac{1}{c} \underset{\sim}{E}_{\perp}' = \gamma\left(\frac{\underset{\sim}{E}_{\perp}}{c} + \underset{\sim}{\beta} \times \underset{\sim}{B}_{\perp}\right)$$

$$\underset{\sim}{B}_{\perp}' = \gamma\left(\underset{\sim}{B}_{\perp} - \underset{\sim}{\beta} \times \frac{\underset{\sim}{E}_{\perp}}{c}\right)$$

For the inverse transformation $\beta \to -\beta$.
Thus, $\underset{\sim}{E}$ and $\underset{\sim}{B}$ don't exist as independent functions

$\Longrightarrow$ they are mixed under the Lorentz transformation

There are, however, limits to the transformation $\underset{\sim}{E} \leftrightarrow \underset{\sim}{B}$ that arise because

$$\frac{1}{c^2} E^2 - B^2$$

is a Lorentz invariant.

The $\frac{1}{c^2}E^2 - B^2$ invariant follows from the product of the field strength tensor with itself,

$$F_{\alpha\beta}F^{\beta\alpha}$$

where $F_{\alpha\beta} = g_{\alpha\gamma}F^{\gamma\delta}g_{\delta\beta}$

$$= \begin{pmatrix} 0 & \frac{1}{c}E_x & \frac{1}{c}E_y & \frac{1}{c}E_z \\ -\frac{1}{c}E_x & 0 & -B_z & B_y \\ -\frac{1}{c}E_y & B_z & 0 & -B_x \\ -\frac{1}{c}E_z & -B_y & B_x & 0 \end{pmatrix}$$

$\Rightarrow$ same as $F^{\alpha\beta}$ but with $\underline{E} \to -\underline{E}$.

Since $F^{\beta\alpha} = (F^{\alpha\beta})^t$,

$$F^{\beta\alpha} = \begin{pmatrix} 0 & \frac{1}{c}E_x & \frac{1}{c}E_y & \frac{1}{c}E_z \\ -\frac{1}{c}E_x & 0 & B_z & -B_y \\ -\frac{1}{c}E_y & -B_z & 0 & B_x \\ -\frac{1}{c}E_z & B_y & -B_x & 0 \end{pmatrix}$$

and

$$F_{\alpha\beta}F^{\beta\alpha} = 2\left(\frac{E^2}{c^2} - B^2\right)$$

We can also evaluated $\frac{E^2}{c^2} - B^2$ in the S and S' systems,

$$\frac{1}{c^2}E'^2 - B'^2 = \frac{1}{c^2}E_{\parallel}^2 - B_{\parallel}^2 + \gamma^2\Big[\frac{1}{c^2}|\underset{\sim}{E}_\perp|^2 + \beta^2|\underset{\sim}{B}_\perp|^2$$

$$+ \cancel{2\,\underset{\sim}{E}_\perp\cdot\underset{\sim}{\beta}\times\underset{\sim}{B}_\perp}\Big] - \gamma^2\Big[|\underset{\sim}{B}_\perp|^2 + \beta^2\frac{1}{c^2}|\underset{\sim}{E}_\perp|^2$$

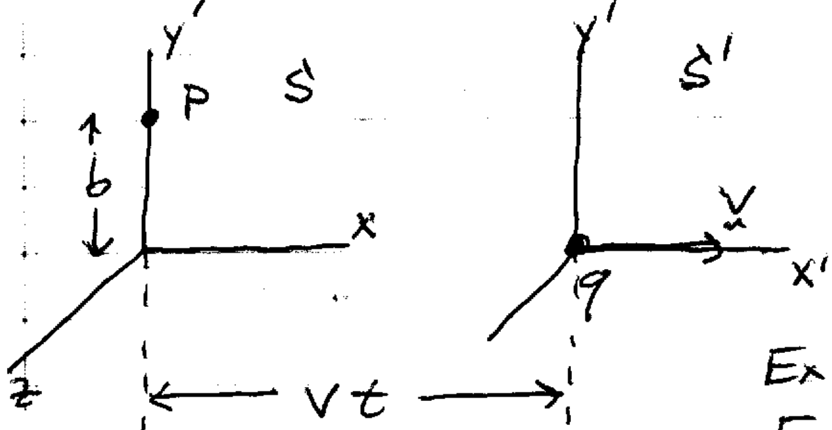
$$- \cancel{2\,\underset{\sim}{B}_\perp\cdot\underset{\sim}{\beta}\times\underset{\sim}{E}_\perp}\Big]$$

$$= \frac{1}{c^2}|\underset{\sim}{E}|^2 - |\underset{\sim}{B}|^2$$

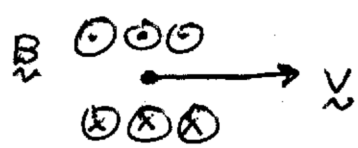
If you have a frame in which there is ~~only~~ only an electric field (no $\underset{\sim}{B}$), you can't transform to a frame in which there is only a magnetic field.

Fields from a moving charge

Consider a point charge at the origin of a moving system S'. An observer is the the point P, a distance b along the y axis of the stationary system S.



Expect P to see E_x, E_y and B_z non-zero



Easiest to evaluate $\underset{\sim}{E}'$, $\underset{\sim}{B}'$ in the S' frame at the position of the observer and then transform those fields to to S frame.

In the S' frame the observer is at

$$x' = -vt'$$
$$y' = b \qquad \Rightarrow r' = \sqrt{b^2 + v^2 t'^2}$$
$$z' = 0$$

In the S' frame there is no magnetic field because the particle is stationary.
At P

$$E_x' = -\frac{q}{r'^2} \frac{vt'}{r'} \frac{1}{4\pi\epsilon_0}$$

$$E_y' = \frac{q}{4\pi\epsilon_0 r'^2} \frac{b}{r'}$$

$$E_z' = 0 \;, \quad \underset{\sim}{B}' = 0$$

First write the fields in terms of space time in S. Have

$$ct' = \gamma(ct - \beta x)$$

At P, $x = 0$ so $t' = \gamma t$.

Thus,

$$E_x' = -\frac{q}{4\pi\epsilon_0}\frac{\gamma v t}{(b^2+\gamma^2 v^2 t^2)^{3/2}}$$

$$E_y' = \frac{q}{4\pi\epsilon_0}\frac{b}{(b^2+\gamma^2 v^2 t^2)^{3/2}}$$

Since E_x, B_x are the same in S and S',

$$E_x = -\frac{q}{4\pi\epsilon_0}\frac{\gamma v t}{(b^2+\gamma^2 v^2 t^2)^{3/2}}, \quad B_x = 0$$

$$\frac{1}{c}E_y = \gamma \frac{1}{c}E_y'$$

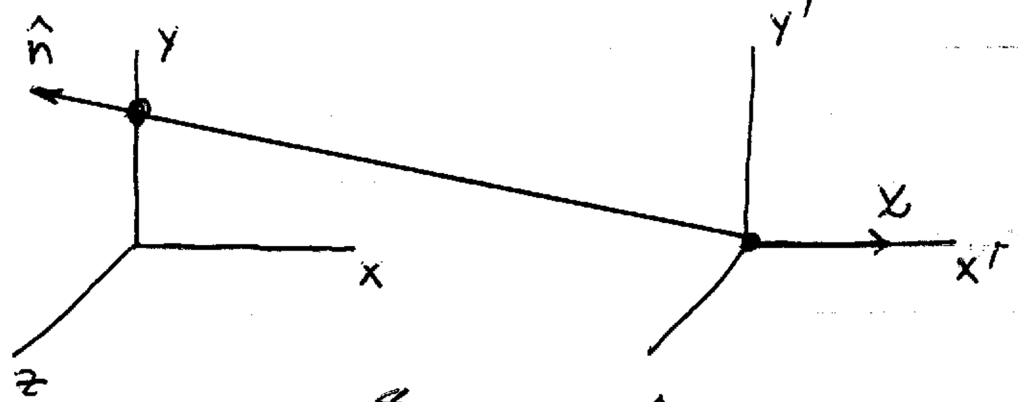
$$E_y = \frac{q}{4\pi\epsilon_0}\frac{\gamma b}{(b^2+\gamma^2 v^2 t^2)^{3/2}}$$

$$B_z = \gamma \beta \frac{E_y'}{c}$$

$$= \frac{q}{4\pi\epsilon_0}\frac{\gamma v b}{c^2(b^2+\gamma^2 v^2 t^2)^{3/2}}$$

$$= \beta\left(\frac{E_y}{c}\right)$$

Non-relativistic limit



$$\underset{\sim}{E} \simeq \frac{q}{4\pi\epsilon_0}\frac{1}{r^2}\hat{n}$$

$$\underset{\sim}{B} = \frac{\mu_0}{4\pi}\int d\underset{\sim}{x}'\,\frac{\underset{\sim}{J}\times\hat{n}}{r^2}$$

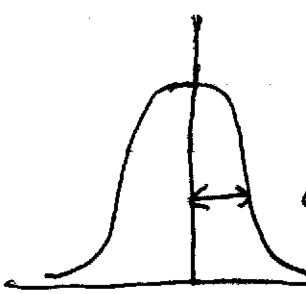
$$\int d\underset{\sim}{x}'\,\underset{\sim}{J} = q\underset{\sim}{V} \quad\Rightarrow\quad \underset{\sim}{B} = q\,\frac{\mu_0\epsilon_0}{4\pi\epsilon_0}\frac{\underset{\sim}{v}\times\hat{n}}{r^2}$$

$$B_z \sim \beta\,\frac{E_y}{c} \qquad = \frac{q}{4\pi\epsilon_0}\,\frac{\underset{\sim}{\beta}\times\hat{n}}{r^2 c}$$

$$\ll \frac{E_y}{c}$$

Strongly relativistic limit

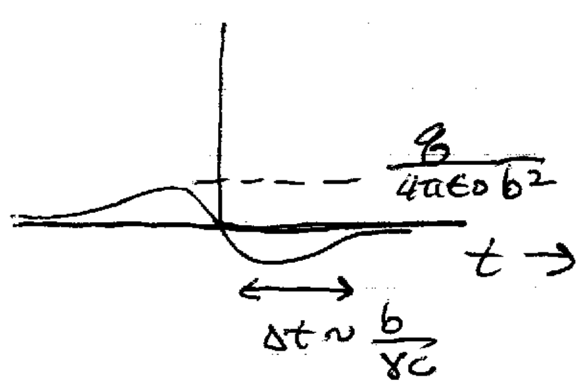
$$B_z \sim \frac{1}{c}E_y$$



$$E_y,\ cB_z \sim \frac{q}{4\pi\epsilon_0 b^2}\,\gamma$$

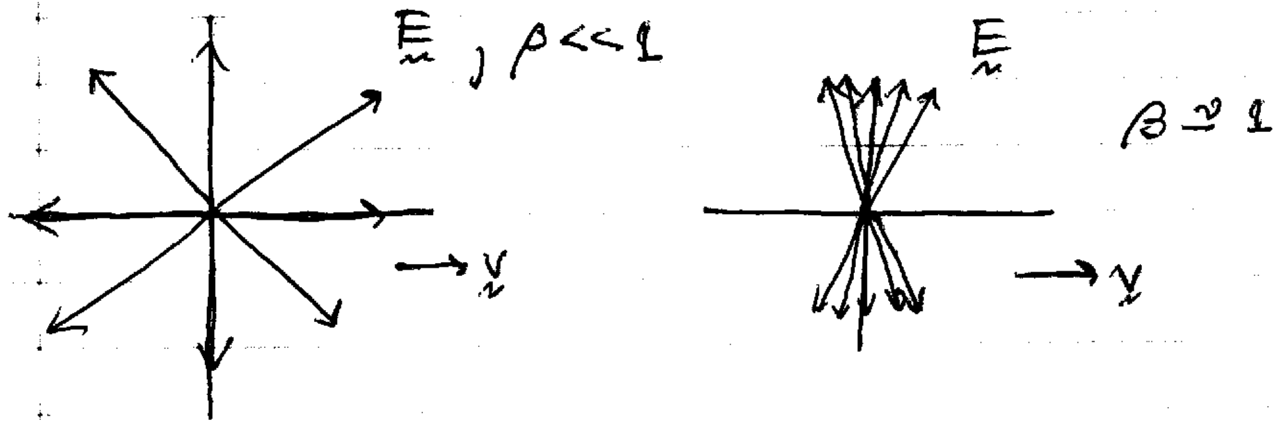
$$\Delta t \sim \frac{b}{\gamma v} \sim \frac{b}{\gamma c}$$

$t \rightarrow$



As γ increases,

E_y and B_z take on larger and larger values over shorter times



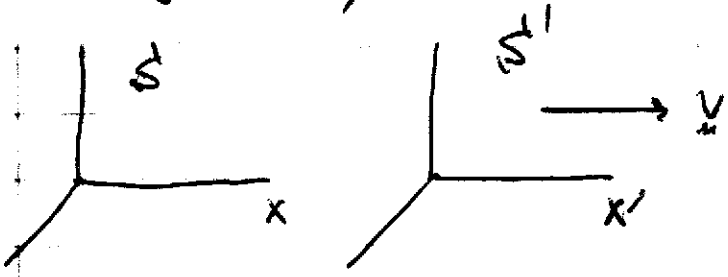
The fields become transverse at high velocity with $\mathbf{E} \perp$ to $\mathbf{B}$, and $\mathbf{E} \times \mathbf{B}$ in the direction of motion

$\Longrightarrow$ acts like a photon

$\Longrightarrow$ space contraction causes $\mathbf{E}$ to twist into the transverse direction

Relativistic Doppler shift

The non-relativistic transformation of ω, k in a Galilean transformation is given by



$$k' = k$$
$$\omega' = \omega - k \cdot V$$

What about under the Lorentz transformation

$\Rightarrow$ the wave phase is an invariant since the phase can be obtained by counting the peaks or troughs of the wave.

Thus, $\quad \phi = \omega t - \underset{\sim}{k} \cdot \underset{\sim}{x} = \omega' t' - \underset{\sim}{k}' \cdot \underset{\sim}{x}'$

$$\omega \gamma \left(t' + \beta \frac{1}{c} x'\right) - k\gamma\left(x' + \beta c t'\right)$$
$$= \omega' t' - k' x'$$
$$= \gamma\left(\omega - kc\beta\right) t' - \gamma\left(k - \frac{\omega}{c}\beta\right) x'$$
$$\omega' = \gamma\left(\omega - kc\beta\right)$$
$$k' = \gamma\left(k - \frac{\omega}{c}\beta\right)$$

$\Rightarrow \left(\frac{\omega}{c}, \underset{\sim}{k}\right)$ form a 4-vector

Also follows from $\partial_\mu \partial^\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - k^2$