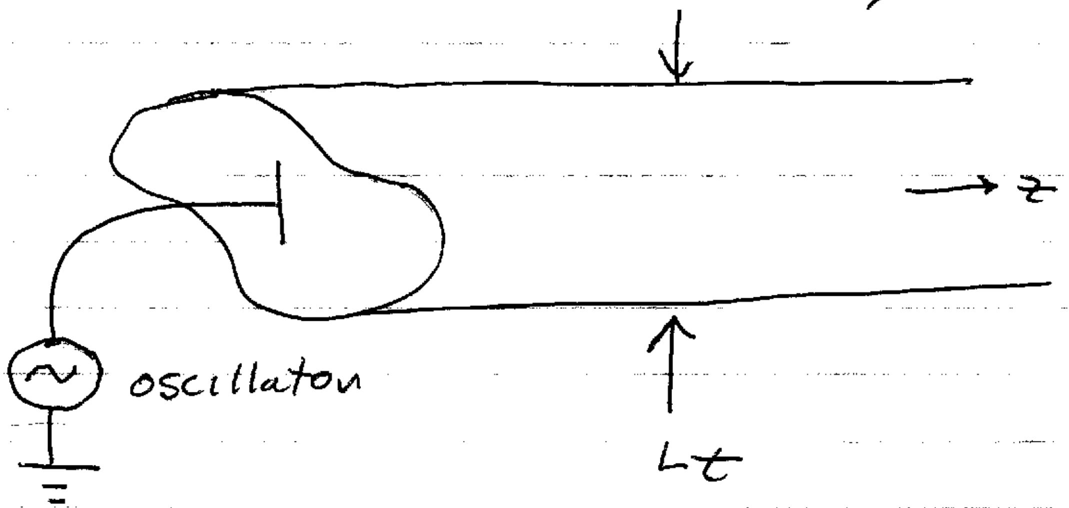
Waveguides

Waveguides are used for transporting electromagnetic wave energy



The antenna oscillates at some frequency ω. For EM waves, need to solve the wave equation inside the guide. The wave equation takes the form

$$\nabla^2 \underline{u} - \frac{\omega^2}{c^2} = \frac{\partial^2}{\partial z^2} + \nabla_t^2$$

$$= \nabla_t^2 - k_z^2$$

where the wave propagates along the guide with a wavevector k_z that needs to be determined.

∇_t^2 comes from solving the equations in the transverse direction and matching the BC's.

Find $\nabla_t^2 \sim -\frac{1}{L_t^2}$ so

$$-\frac{\omega^2}{c^2} \simeq -\frac{1}{L_t^2} - k_z^2$$

or

$$k_z^2 \simeq \frac{\omega^2}{c^2} - \frac{1}{L_t^2}$$

The wave can propagate as long as $k_z^2 > 0$ or

$$\frac{\omega^2}{c^2} > \frac{1}{L_t^2}$$

Below the cutoff frequency $\omega_c \sim c/L_t$ there is no propagation.

Ideal waveguides $\Rightarrow$ ideal conducting boundaries

Assume $\underset{\sim}{B}, \underset{\sim}{E} \sim e^{-i\omega t} e^{\pm i k z}$

Interior to the guide have

$$\nabla \times \underset{\sim}{E} = i\omega \underset{\sim}{B}, \quad \nabla \cdot \underset{\sim}{B} = 0$$

$$\nabla \times \underset{\sim}{B} = -i\omega \mu_0 \epsilon_0 \underset{\sim}{E}, \quad \nabla \cdot \underset{\sim}{E} = 0$$

$$\Rightarrow \left(\nabla^2 + \mu_0 \epsilon_0 \omega^2\right)\begin{pmatrix} \underset{\sim}{E} \\ \underset{\sim}{B} \end{pmatrix} = 0$$

$$\Rightarrow \left(\nabla_t^2 + \mu_0 \epsilon_0 \omega^2 - k^2\right)\begin{pmatrix} \underset{\sim}{E} \\ \underset{\sim}{B} \end{pmatrix} = 0$$

$$\underline{E} = \underline{E}_t + \underline{E}_z$$

Separate $\underline{B}$ and $\underline{E}$ into axial and transverse components. Define class of guide mode by the presence or absence of E_z, B_z.

Three basic mode types:

1. (TEM) Transverse Electric and Magnetic

   $B_z, E_z = 0$ everywhere

2. (TM) Transverse magnetic

   ~~$\Rightarrow B_z = 0$ but $E_z \neq 0$~~

3. (TE) Transverse electric

First sketch picture of basic mode structure. For TE, TM modes solve the wave eqn for E_z or B_z with appropriate BCs

$$\left(\nabla_t^2 + \mu\epsilon\omega^2 - k^2\right)\begin{pmatrix} E_z \\ B_z \end{pmatrix} = 0$$

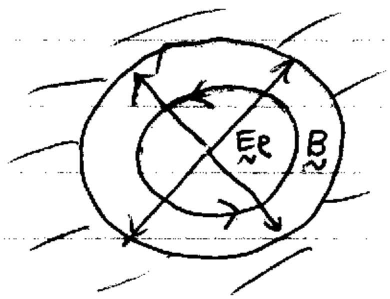
$\Rightarrow$ Solve for other components using MEs.

For TEM use MEs with $B_z = E_z = 0$.

TEM modes $\quad E_z, B_z = 0$

Can a TEM mode exist in a simple cylindrical guide?

Since $\nabla \cdot \hat{\mathbf{B}} = 0$ and $\mathbf{B} \cdot \hat{n} = 0$, lines of $\mathbf{B}$ must close on themselves



$\mathbf{B}$ must be azimuthal. From Faraday's law

$$\nabla \times \mathbf{E} - i\omega \mathbf{B} = 0$$

Operate with $\hat{z} \times$

$$\hat{z} \times (\nabla \times \mathbf{E}) = \cancel{\nabla E_z}^{\,0} - \frac{\partial}{\partial z}\mathbf{E} = -\frac{\partial}{\partial z}\mathbf{E}_t$$

$$= i\omega\, \hat{z} \times \mathbf{B}_t$$

$$\mathbf{E}_t = -\frac{\omega}{K}\, \hat{z} \times \mathbf{B}_t$$

$\Rightarrow \mathbf{E}_t$ is radial

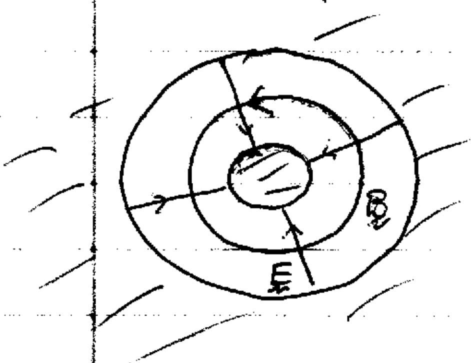
But

$$\nabla \cdot \mathbf{E} = 0 \Rightarrow \frac{1}{\rho}\frac{\partial}{\partial \rho}\rho E_\rho = 0$$

$\Rightarrow \rho E_\rho = \text{const.}$ since E_ρ must be bounded at $\rho = 0$.

$\Rightarrow$ no TEM modes in a simple guide

What about a guide with a conducting core ?



Wave with B_φ, E_ρ ?

$$\nabla \cdot \underset{\sim}{E} = 0 \Rightarrow \frac{1}{\rho}\frac{\partial}{\partial \rho}\rho E_\rho = 0$$

$$E_\rho \sim \frac{1}{\rho} \Rightarrow \text{ok}$$

since $\rho \neq 0$.

$$\nabla \times \underset{\sim}{E} = i\omega \underset{\sim}{B} \qquad \nabla \times \underset{\sim}{B} = \mu_0\epsilon_0(-i\omega)\underset{\sim}{E}$$

$$(\nabla \times \underset{\sim}{E})_\varphi = i\omega B_\varphi \qquad (\nabla \times B)_\rho = \frac{1}{c^2}(-i\omega)E_\rho$$

$$\frac{\partial}{\partial z}E_\rho = i\omega B_\varphi \qquad -ikB_\varphi = \frac{1}{c^2}(-i\omega)E_\rho$$

$$kE_\rho = \omega B_\varphi \qquad kB_\varphi = \frac{\omega}{c^2}E_\rho$$

$$\Rightarrow \quad k\left(\frac{kc^2}{\omega}\right)B_\varphi = \omega B_\varphi$$

$$k^2 = \frac{\omega^2}{c^2} \Rightarrow$$ same as free space dispersion relation

$\Rightarrow$ TEM modes don't have a cutoff frequency

$\Rightarrow$ wave equation does not involve ∇_t.

Poynting vector for TEM

$$\underset{\sim}{\bar{S}} \cdot \hat{z} = \frac{1}{2} \hat{z} \cdot E_0 \times B_0^* \frac{1}{\mu_0}$$

$$= \frac{1}{2} E_{0\ell} B_{0\ell}^* \frac{1}{\mu_0}$$

$$= \frac{1}{2} \frac{\omega}{k} |B_{0\ell}|^2 \frac{1}{\mu_0}$$

$$= c \, \frac{1}{2} \frac{1}{\mu_0} |B_{0\ell}|^2 = c \, \bar{u}$$

with $\bar{u} = \frac{1}{2} \frac{1}{\mu_0} |B_{0\ell}|^2$ the wave energy density

$\Rightarrow$ recall that

$$\bar{u} = \frac{1}{4} \left(\frac{1}{\mu_0} |B_{0\ell}|^2 + \epsilon_0 |E_{0\ell}|^2 \right)$$

with

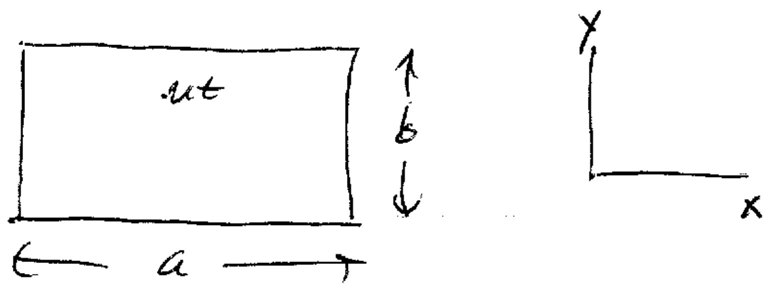
$$\frac{1}{\mu_0} |B_{0\ell}|^2 = \epsilon_0 |E_{0\ell}|^2$$

Transverse Electric (TE) and Magnetic (TM) Waves

TE waves $\Longrightarrow$ $E_z = 0$ everywhere

TM waves $\Longrightarrow$ $B_z = 0$ everywhere

Consider TE mode in rectangular waveguide



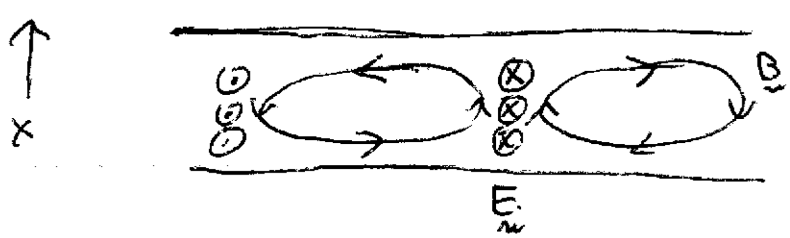
$a > b$

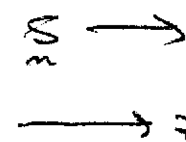
$E_z = 0$ everywhere

$$\left(\nabla_t^2 + \mu\epsilon\omega^2 - k^2\right) B_z = 0$$

What is lowest order mode $\Rightarrow$ want small ∇_T

$$\Longrightarrow \frac{\partial}{\partial y} = 0$$





Have B_z, B_x, E_y

sign of $\underset{\sim}{E}$ from $S_z > 0$

Boundary conditions?

$E_y = 0$ at $x = 0, a$

$B_x = 0$

$$\left[\nabla \times \underset{\sim}{B} = -\frac{i\omega}{\cancel{c}}\mu\epsilon\,\underset{\sim}{E}\right]$$

$$ikB_x - \frac{\partial}{\partial x}B_z = -\frac{i\omega}{c}\mu\epsilon E_y$$

$$\Longrightarrow \boxed{\frac{\partial}{\partial x}B_z = 0} \quad \text{at } x = 0, a$$

general B.C. $\hat{n}\cdot\nabla B_z = 0$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\mu\epsilon\omega^2}{\cancel{c^2}} - k^2\right)B_z = 0$$

$$\boxed{B_z = B_{z0}\,e^{-i\omega t}\,e^{ikz}\cos\left(\frac{m\pi x}{a}\right)}$$

$m = 1, 2, 3 \ldots$

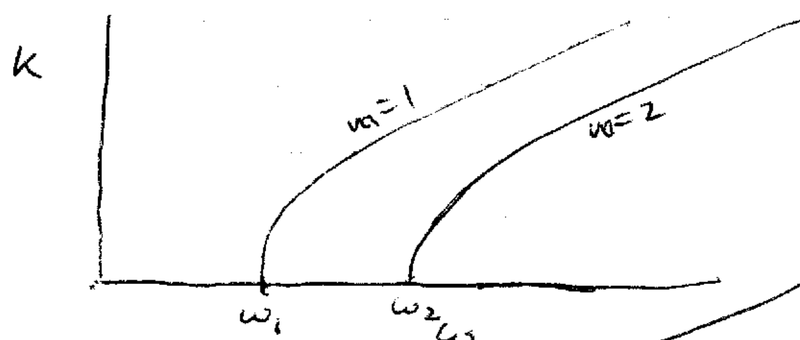
$$\cancel{\frac{\partial}{\partial x}B_x + ikB_z = 0 \Rightarrow B_x}$$

$$\boxed{k^2 = \frac{\omega^2}{\cancel{c^2}}\mu\epsilon - \frac{m^2\pi^2}{a^2}}$$

$$\cancel{2}k = \frac{\cancel{2}\omega}{\cancel{c^2}}\frac{n^2}{c^2}V_g$$

$$\boxed{V_g = \frac{kc^2}{\omega n^2}}$$

$V_g \to 0$ as $\omega \to \omega_1$



$k_{free} = \frac{\omega n}{c} > k_x = \frac{\pi}{a}$ to have propagation

$$\boxed{\omega_1 = \frac{\pi c}{a n}}$$

cutoff frequency

$\Longrightarrow$ no propagation below ω_1

$\Longrightarrow$ select frequency so only single mode propagates

<u>Other components</u>

$$\frac{\partial}{\partial x} B_x + ik B_z = 0$$

$$\frac{\partial}{\partial x} B_x = -ik B_{z0} e^{-i\omega t} e^{ikz} \cos\left(\frac{m\pi x}{a}\right)$$

$$B_{x0} = -ik B_{z0} \left(\sin \frac{m\pi x}{a}\right) \frac{a}{m\pi}$$

$$\frac{\partial}{\partial t} \underset{\sim}{B} + \nabla \times \underset{\sim}{E} = 0$$

$$-i\omega B_{x0} + (-ik E_{y0}) = 0$$

$$E_{y0} = -\frac{\omega}{k} B_{x0} \quad \Rightarrow \quad E_y \sim \sin\left(\frac{m\pi x}{a}\right)$$

<u>Poynting flux yields energy flux down guide</u> $\Rightarrow P$

$$P = \int dx\,dy\ \bar{S}_z = \frac{1}{2} \int dx\,dy \left(-E_{y0} B^*_{x0}\right) \sin^2\left(\frac{m\pi x}{a}\right) \frac{1}{\mu_0}$$

$$= \frac{1}{4} \frac{ab}{\mu_0} \frac{\omega}{k} |B_{x0}|^2 = \frac{1}{4} ab \frac{\omega}{k} k^2 \frac{|B_{z0}|^2}{\mu_0} \frac{a^2}{\pi^2}$$

average energy

$$\int dx\,dy\ \bar{u} = \frac{1}{2}\left(\frac{1}{2}\epsilon_0 |E_{y0}|^2 + \frac{1}{2\mu_0}\left(|B_{x0}|^2 + |B_{z0}|^2\right)\right) \frac{1}{2} ab$$

$$= \frac{1}{8}\left(\left(\epsilon_0 \frac{\omega^2}{k^2} + \frac{1}{\mu_0}\right) B_{x0}^2 + B_{z0}^2 \frac{1}{\mu_0}\right) ab$$

$$\cancel{\frac{1}{8\mu_0}\left[\frac{\pi^2}{a^2} k^2 |B_{z0}|^2 \frac{a^2}{\pi^2} + |B_{z0}|^2\right]}$$

$$= \frac{1}{8\mu_0}\left[\left(2 + \frac{\pi^2}{k^2 a^2}\right) k^2 \frac{a^2}{\pi^2} + 1\right] |B_{z0}|^2\, ab$$

$$\int dx\,dy\,\bar{u} = \frac{|B_{z0}^2|}{8\mu_0}\,2\left(1+\frac{k^2a^2}{\pi^2}\right)ab$$

$$= \frac{|B_{z0}^2|}{4\mu_0}\left(\frac{\omega^2\mu_0\epsilon_0 a^2}{\pi^2}\right)ab$$

$$\int dx\,dy\,\bar{S}_z = \frac{1}{\omega}\frac{1}{\mu_0\epsilon_0}k\int dx\,dy\,\bar{u}$$

$$\cancel{2}k = \cancel{2}\omega V_g \mu_0\epsilon_0 \Rightarrow V_g = \frac{k}{\omega}\frac{1}{\mu_0\epsilon_0}$$

$$V_g = \frac{d\omega}{dk}$$

$$P = \int dx\,dy\,\bar{S}_z = V_g\int dx\,dy\,\bar{u}$$

$\Rightarrow$ the energy propagates down the guide at the group velocity

Energy absorption by the walls

$\Rightarrow$ retain finite conductivity

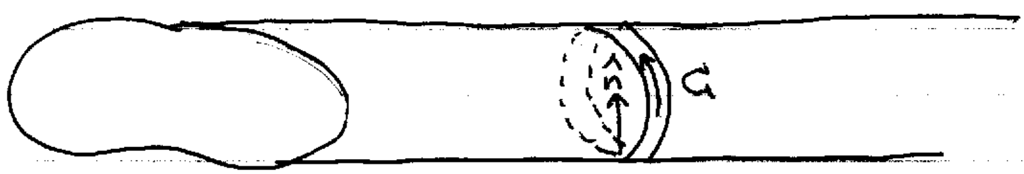
$\Rightarrow$ small imaginary contribution to k

$$k \Rightarrow k + i k_I$$

$$P \sim e^{-2k_I z}$$

$$-\frac{dP}{dz} = 2k_I P = \frac{\text{energy loss rate}}{\text{unit distance along } z}$$

$$= -\oint_C dl\,\bar{S}\cdot\hat{n}$$



Previously calculated the energy flux into the conductor

$$\Rightarrow -\hat{n}\cdot \bar{S} = \frac{\omega\delta}{4\mu_0}|B_t|^2$$

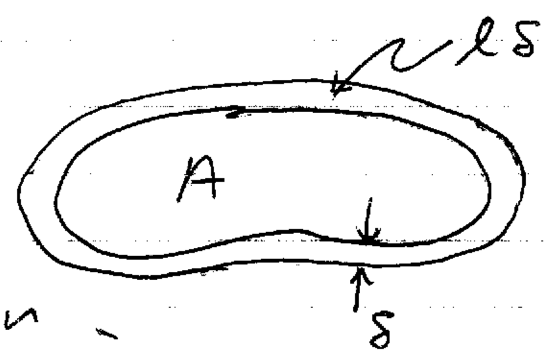
$$K_I = \frac{1}{8P}\,\frac{\omega\delta}{\mu_0}\oint_G dl\,|B_t|^2$$

$$P \approx \frac{1}{2\mu_0}\,\hat{z}\cdot\left(E_{t0}\times B^*_{t0}\right)A$$

$$\sim \frac{1}{2\mu_0}\,\frac{\omega|B|^2}{K}A$$

$$K_I \sim \frac{\omega\delta}{\mu_0}\,\frac{l\,|B|^2}{\frac{\omega}{K}|B|^2\frac{1}{\mu_0}A} \sim K\,\frac{\delta l}{A}$$

$$\frac{K_I}{K} \sim \frac{\delta l}{A}$$

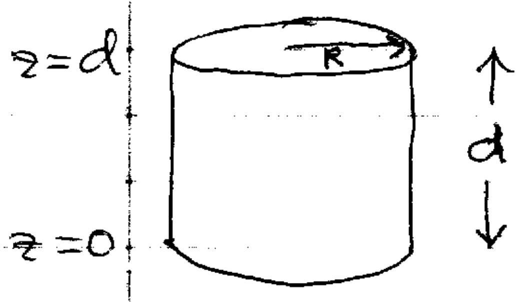


$$\sim \frac{\text{area with power inside conductor}}{\text{interior area of guide}}$$

Right circular resonance cavity

Waveguides are designed to transport EM wave energy. A resonance cavity has caps at each end to reflect the waves.

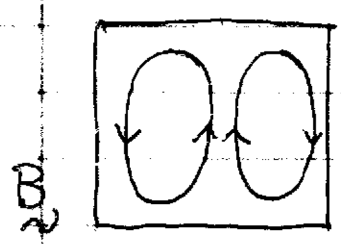
Consider a right circular cylinder of radius R and length d.



Investigate TE waves in this cavity.

$$\Rightarrow E_z = 0$$

$B_z \neq 0$ but $B_z = 0$ at $z = 0, d$.



$m=0$ mode

$$\Rightarrow B_\rho \neq 0$$

$$B_z = B_{z0}(\rho, \varphi)\, \sin\left(\frac{p\pi z}{d}\right)$$

$$\nabla_t^2 B_{z0} + \underbrace{\left(\frac{\mu\epsilon\omega^2}{c^2} - k^2\right)}_{\gamma^2} B_{z0} = 0$$

$$\frac{\partial}{\partial \rho} B_{z0} \Big|_{\rho=R} = 0 \quad \Rightarrow \text{same as waveguide}$$

$$k = \frac{p\pi}{d} \qquad \frac{\mu\epsilon\omega^2}{c^2} - k^2 = \gamma^2$$

$$B_{z0} = B_0 J_m(\gamma\rho) e^{\pm im\varphi}$$

$$\Rightarrow \text{B.C.} \quad J_m'(\gamma R) = 0$$

$$\gamma R = x'_{mn} \qquad \Rightarrow \gamma = \frac{x'_{mn}}{R}$$

$$J_m'(x'_{mn}) = 0$$

x'_{mn} are tabulated

$$\frac{\mu\epsilon\omega^2}{c^2} - k^2 = \frac{x'^2_{mn}}{R^2}$$

x'_{mn} smallest for $m=1$

$x'_{11} = 1.841$

$$\boxed{\omega^2 = \left(\frac{p^2\pi^2}{d^2} + \frac{x'^2_{mn}}{R^2}\right)\frac{c^2}{\mu\epsilon}}$$

~~p, m, n integers~~ $m = 0, 1, 2 \ldots$

$p, n = 1, 2, 3 \ldots$

$\Rightarrow$ resonant frequencies of the cavity.

Q of a cavity

Let $\bar{U}$ = total stored energy in a cavity. Due to Joule heating of the walls, the energy will decrease with time,

$$\frac{1}{\bar{U}}\frac{d}{dt}\bar{U} \equiv -\frac{\omega}{Q}$$

$$\Rightarrow \bar{U} \sim e^{-\frac{\omega}{Q}t}$$

with ω the cavity resonant frequency.

$$\text{large } Q \Rightarrow \text{small energy loss rate}$$

Poynting flux into walls

$$-\bar{S}\cdot\hat{n} \sim \frac{|B_\parallel|^2}{4\mu_0} c\, k\delta$$

$\Rightarrow$ integrate over surface area A.

$$\text{rate of power loss} \sim \frac{|B_\parallel|^2}{4\mu_0}\,\omega A\delta \sim \frac{d}{dt}\bar{U}$$

$$\bar{U} \sim \frac{B^2}{4\mu_0} V \quad \Rightarrow \bar{V} = \text{cavity volume}$$

$$\omega A\delta \sim \frac{\omega}{Q} V \quad \Rightarrow Q = \frac{\bar{V}}{A\delta} = \frac{\text{vol. cavity}}{\text{Vol in conductor}}$$

$$\Rightarrow Q \text{ large}$$

Resonant width of a cavity

The dissipation in a cavity produces a finite band width of the resonant frequency of the cavity. The lifetime of the energy

$$\tau \sim \frac{Q}{\omega}$$

The cavity cannot sense a frequency shift $\Delta\omega$ smaller than

$$\Delta\omega\,\tau \sim \frac{\Delta\omega Q}{\omega} < 1$$

or

$$\frac{\Delta\omega}{\omega} \lesssim \frac{1}{Q}$$

$\Rightarrow$ this is the cavity band width.

$\Rightarrow$ Within $\Delta\omega$ of ω the cavity can reach its maximum energy density.