Maxwell's Equations

We write the coupled, time-dependent equations for $\underset{\sim}{E}, \underset{\sim}{B}$ as

$$\nabla \cdot \underset{\sim}{D} = \rho \qquad \underset{\sim}{D} = \epsilon \underset{\sim}{E}$$

$$\nabla \cdot \underset{\sim}{B} = 0 \qquad \underset{\sim}{B} = \mu \underset{\sim}{H}$$

$$\nabla \times \underset{\sim}{E} + \frac{\partial}{\partial t} \underset{\sim}{B} = 0$$

$$\nabla \times \underset{\sim}{H} = \underset{\sim}{J} + \frac{\partial}{\partial t} \underset{\sim}{D}$$

Have included an additional term

$$\frac{\partial}{\partial t} \underset{\sim}{D} \equiv \text{displacement current}$$

in the $\nabla \times \underset{\sim}{H}$ eqn. In a time-dependent situation this term is required. Consider

$$\nabla \cdot (\nabla \times \underset{\sim}{H}) = 0 = \nabla \cdot \underset{\sim}{J} + \frac{\partial}{\partial t} \nabla \cdot \underset{\sim}{D}$$

but from charge conservation

$$\frac{\partial}{\partial t} \rho + \nabla \cdot \underset{\sim}{J} = 0$$

This yields

$$0 = -\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial t}\nabla\cdot\underline{D}$$

or

$$0 = -\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial t}\rho$$

Thus, the displacement current is required to maintain charge conservation.

Vector and scalar potentials

Instead of directly solving Maxwell's eqns. for $\underline{E}, \underline{B}$, it is often convenient to solve the eqns for the vector potential $\underline{A}$ and scalar potential φ.

Since $\nabla\cdot\underline{B} = 0 \implies \underline{B} = \nabla\times\underline{A}$

From Faraday's law

$$\Rightarrow \nabla\times\underline{E} + \frac{\partial}{\partial t}\nabla\times\underline{A} = 0$$

or

$$\nabla\times(\underbrace{\underline{E} + \frac{\partial}{\partial t}\underline{A}}_{-\nabla\varphi}) = 0$$

Thus, $\underset{\sim}{E} = -\nabla\varphi - \frac{\partial}{\partial t}\underset{\sim}{A}$

where φ is the usual scalar potential. Can re-write Maxwell's eqns in terms of $\underset{\sim}{A}$, φ. For simplicity consider only free space, $\mu = \mu_0$, $\epsilon = \epsilon_0$

$$\nabla\cdot\underset{\sim}{E} = \frac{\rho}{\epsilon_0} = -\nabla^2\varphi - \frac{\partial}{\partial t}\nabla\cdot\underset{\sim}{A}$$

$$\Rightarrow \boxed{\nabla^2\varphi + \frac{\partial}{\partial t}\nabla\cdot\underset{\sim}{A} = -\frac{\rho}{\epsilon_0}}$$

$$\nabla\times\underset{\sim}{B} = \nabla\times(\nabla\times\underset{\sim}{A}) = \nabla(\nabla\cdot\underset{\sim}{A}) - \nabla^2\underset{\sim}{A}$$

$$= \mu_0\left[\underset{\sim}{J} - \epsilon_0\nabla\frac{\partial\varphi}{\partial t} - \epsilon_0\frac{\partial^2\underset{\sim}{A}}{\partial t^2}\right]$$

or

$$\boxed{\frac{1}{c^2}\frac{\partial^2\underset{\sim}{A}}{\partial t^2} - \nabla^2\underset{\sim}{A} + \nabla\left(\nabla\cdot\underset{\sim}{A} + \frac{1}{c^2}\frac{\partial}{\partial t}\varphi\right) = \mu_0\underset{\sim}{J}}$$

where $c^2 = 1/\mu_0\epsilon_0$

$\Rightarrow$ c is the velocity of light

$= 3\times 10^8$ m/s

Maxwell's eqns are now reduced to two coupled equations for $\underset{\sim}{A}$, φ.

However, the solutions for $\underset{\sim}{A}$ and φ are not unique. Let

$$A \longrightarrow A + \nabla \Lambda \quad \Rightarrow \text{leaves } \underset{\sim}{B} \text{ unchanged}$$

$$\varphi \longrightarrow \varphi - \frac{\partial}{\partial t}\Lambda \quad \Rightarrow \text{leaves } \underset{\sim}{E} \text{ unchanged}$$

This transformation also leaves the equations for $\underset{\sim}{A}$ and φ unchanged

$\Rightarrow$ Guage transformation

$\Rightarrow$ Allows us to decouple the eqns for $\underset{\sim}{A}$ and φ

Lorentz guage

Choose

$$\nabla \cdot \underset{\sim}{A} + \frac{1}{c^2}\frac{\partial \varphi}{\partial t} = 0 \qquad \text{Lorentz condition}$$

Can accomplish this by the choice of Λ in the guage transformation.

Suppose $\nabla \cdot \underset{\sim}{A} + \frac{1}{c^2}\frac{\partial}{\partial t}\varphi \neq 0$, then choose

$$\underset{\sim}{A}' = A + \nabla \Lambda$$

$$\varphi' = \varphi - \frac{\partial}{\partial t}\Lambda$$

and

$$\nabla \cdot \underset{\sim}{A}' + \frac{1}{c^2}\frac{\partial}{\partial t}\varphi' = \nabla \cdot \underset{\sim}{A} + \nabla^2 \Lambda + \frac{1}{c^2}\frac{\partial}{\partial t}\varphi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Lambda$$

$$= 0$$

is satisfied for

$$-\nabla^2 \Lambda + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Lambda = \nabla \cdot \underset{\sim}{A} + \frac{1}{c^2}\frac{\partial}{\partial t}\varphi$$

We will show later that this eqn can always be solved using a Green's function for the wave eqn.

The potentials that satisfy the Lorentz condition belong to the Lorentz guage

$$\Longrightarrow \underset{\sim}{A}, \varphi \text{ decouple}$$

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\underset{\sim}{A} - \nabla^2 \underset{\sim}{A} = \mu_0 \underset{\sim}{J}$$

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\varphi - \nabla^2 \varphi = \frac{1}{\epsilon_0}\rho$$

These take the form of wave equations driven by $\underset{\sim}{J}$, ρ.

$\Rightarrow$ wave speed is c in a vacuum.

Coulomb guage

The Coulomb guage is defined by the choice

$$\nabla \cdot \underset{\sim}{A} = 0,$$

which yields

$$\nabla^2 \varphi = -\frac{\rho}{\epsilon_0}$$

In this case φ is the usual Coulomb potential of electrostatics. Also have

$$\nabla^2 \underset{\sim}{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\underset{\sim}{A} = -\mu_0 \underset{\sim}{J} + \frac{1}{c^2}\nabla \frac{\partial}{\partial t}\varphi$$

$\Rightarrow$ φ and $\underset{\sim}{A}$ no longer decouple

Poyntings theorem and energy conservation

Consider a medium defined by time stationary μ, ϵ in which have time-dependent $\underset{\sim}{E}, \underset{\sim}{B}$. We previous showed that the energy density in $\underset{\sim}{E}$ and $\underset{\sim}{B}$ is given by

$$u = \frac{1}{2}\left(\underset{\sim}{E}\cdot\underset{\sim}{D} + \underset{\sim}{H}\cdot\underset{\sim}{B}\right)$$

$$= \frac{1}{2}\left(\epsilon|\underset{\sim}{E}|^2 + \frac{1}{\mu}|\underset{\sim}{B}|^2\right)$$

with $\underset{\sim}{D} = \epsilon\underset{\sim}{E}$ and $\underset{\sim}{B} = \mu\underset{\sim}{H}$. Want to explore whether this energy density is consistent with the full set of Maxwell's equations.

$\Rightarrow$ need to establish a conservation law for energy

Consider

$$\frac{\partial}{\partial t}u = \underset{\sim}{E}\cdot\frac{\partial}{\partial t}\underset{\sim}{D} + \underset{\sim}{H}\cdot\frac{\partial}{\partial t}\underset{\sim}{B}$$

$$= \underset{\sim}{E}\cdot\left(\nabla\times\underset{\sim}{H} - \underset{\sim}{J}\right) + \underset{\sim}{H}\cdot\left(-\nabla\times\underset{\sim}{E}\right)$$

$$= -\underset{\sim}{J}\cdot\underset{\sim}{E} + \underbrace{\left(\underset{\sim}{E}\cdot\nabla\times\underset{\sim}{H} - \underset{\sim}{H}\cdot\nabla\times\underset{\sim}{E}\right)}_{-\nabla\cdot\left(\underset{\sim}{E}\times\underset{\sim}{H}\right)}$$

$$\frac{\partial u}{\partial t} + \nabla \cdot (\mathbf{E} \times \mathbf{H}) = -\mathbf{J} \cdot \mathbf{E}$$

$\mathbf{S} = \mathbf{E} \times \mathbf{H} \equiv$ Poynting vector

$\Rightarrow$ flux of electromagnetic energy

What are the units for S?

$$S \sim EH \approx (\epsilon^{1/2} E)\left(\frac{1}{\mu^{1/2}} B\right) \frac{1}{\epsilon^{1/2} \mu^{1/2}}$$

$$\sim \frac{\text{energy}}{\text{vol}} \text{ velocity}$$

$$\sim \frac{\text{energy}}{\text{area time}}$$

What about $\mathbf{J} \cdot \mathbf{E}$? Consider a small volume element $d\mathbf{x}$,

$$d\mathbf{x}\, \mathbf{J} \cdot \mathbf{E} = \sum d\mathbf{x}\, n e \mathbf{v} \cdot \mathbf{E}$$

$$= \sum_j e_j \mathbf{v}_j \cdot \mathbf{E} = \sum_j \mathbf{F}_j \cdot \mathbf{v}_j$$

where $\sum_j$ is a sum over all the charges in volume $d\mathbf{x}$ and $\mathbf{F}_j$ is the force on the charge j.

$\Rightarrow \mathbf{F}_j \cdot \mathbf{v}_j$ is the rate work done on the jth charge.

Note that $\underset{\sim}{B}$ does not do work on charges

$$\Longrightarrow \underset{\sim}{v}_j \cdot (\underset{\sim}{v}_i \times \underset{\sim}{B}) = 0$$

$\underset{\sim}{J} \cdot \underset{\sim}{E}$ = $\dfrac{\text{rate work is done on charges}}{\text{unit volume}}$

$-\underset{\sim}{J} \cdot \underset{\sim}{E}$ = $\dfrac{\text{rate at which electromagnetic field energy is transferred to charges}}{\text{unit volume}}$

Thus,

the rate of change of local energy density = − divergence of electromagnetic energy flux

− local charge heating

$\Longrightarrow$ Conservation of energy

Electromagnetic radiation from a static charge and current distribution

The existence of the Poynting flux $\underset{\sim}{S} = \underset{\sim}{E} \times \underset{\sim}{H}$ raises the question about where static distributions of charge and current are able to radiate. We want to demonstrate that they don't.

Consider localized current and charge distributions that are time independent.

$$\text{Energy radiated} = \int_V d\underset{\sim}{x}\; \nabla \cdot (\underset{\sim}{E} \times \underset{\sim}{H})$$

$$= \int d\underset{\sim}{x}\; (\underset{\sim}{H} \cdot \nabla \times \underset{\sim}{E} - \underset{\sim}{E} \cdot \nabla \times \underset{\sim}{H})$$

$$= \int d\underset{\sim}{x}\; \Big(\cancelto{0}{-\underset{\sim}{H} \cdot \frac{\partial}{\partial t}\underset{\sim}{B}} - \underset{\sim}{E} \cdot \underset{\sim}{J} - \cancelto{0}{\underset{\sim}{E} \cdot \frac{\partial \underset{\sim}{D}}{\partial t}}\Big)$$

$$= -\int d\underset{\sim}{x}\; \underset{\sim}{J} \cdot \underset{\sim}{E} = \int d\underset{\sim}{x}\; \underset{\sim}{J} \cdot \nabla \varphi$$

$$= \int d\underset{\sim}{x}\; \varphi\, \nabla \cdot \underset{\sim}{J} = 0$$

$\Rightarrow$ no radiated energy

Momentum conservation

Want to write an equation that includes the total momentum of fields and particles ~~to verify~~ that descibes how momentum is transferred and conserved.

Consider the ~~force~~ on a particle

$$\frac{d}{dt} \underset{\sim}{P}_j = m_j \frac{d}{dt} \underset{\sim}{v}_j = q_j \underset{\sim}{E}(\underset{\sim}{x}_j) + q_j \underset{\sim}{v}_j \times \underset{\sim}{B}(x_j)$$

Sum over all particles in an infinitesimal volume $d\underset{\sim}{x}$ and assume $\underset{\sim}{E}(\underset{\sim}{x}_j) \simeq \underset{\sim}{E}(\underset{\sim}{x})$

$$\sum_j \frac{q_j}{d\underset{\sim}{x}} = \rho(\underset{\sim}{x}) \quad , \quad \sum_j \frac{q_j v_j}{d\underset{\sim}{x}} = \underset{\sim}{J}(\underset{\sim}{x})$$

$$\underset{\sim}{P} = \sum_j \underset{\sim}{P}_j \frac{1}{d\underset{\sim}{x}} = \text{momentum density}$$

$$\frac{d}{dt} \underset{\sim}{P} = \rho \underset{\sim}{E} + \underset{\sim}{J} \times \underset{\sim}{B}$$

Integrate over volume

$$\underset{\sim}{P}_{mech} \equiv \int d\underset{\sim}{x} \; \underset{\sim}{P}(\underset{\sim}{x})$$

$$= \text{particle momentum}$$

$$\frac{d}{dt}\underset{\sim}{P}_{mech} = \int d\underset{\sim}{x} \; (\rho \underset{\sim}{E} + \underset{\sim}{J} \times \underset{\sim}{B})$$

Have

$$\rho = \epsilon_0 \nabla \cdot \underset{\sim}{E} \;, \quad \underset{\sim}{J} = \frac{1}{\mu_0} \nabla \times \underset{\sim}{B} - \epsilon_0 \frac{\partial}{\partial t} \underset{\sim}{E}$$

so

$$\frac{d}{dt}\underset{\sim}{P}_{mech} = \int d\underset{\sim}{x} \Big[\epsilon_0 \underset{\sim}{E} \, \nabla \cdot \underset{\sim}{E} + \frac{1}{\mu_0} (\nabla \times \underset{\sim}{B}) \times \underset{\sim}{B}$$

$$- \epsilon_0 \underbrace{\frac{\partial}{\partial t} \underset{\sim}{E} \times \underset{\sim}{B}}_{\underbrace{\frac{\partial}{\partial t} \underset{\sim}{E} \times \underset{\sim}{B}}_{\mu_0 \frac{\partial}{\partial t} \underset{\sim}{S}} - \underset{\sim}{E} \times \frac{\partial}{\partial t} \underset{\sim}{B}} \Big]$$

The integral over $\underset{\sim}{S}$ is the total ~~wave~~ electromagnetic momentum

$$\underset{\sim}{P}_{em} = \frac{1}{c^2} \int d\underset{\sim}{x} \; \underset{\sim}{E} \times \underset{\sim}{H} = \frac{1}{c^2} \int d\underset{\sim}{x} \; \underset{\sim}{S}$$

$$\frac{d}{dt}\left(\underset{\sim}{P}_{mech} + \underset{\sim}{P}_{em} \right) = \int d\underset{\sim}{x} \Big[\epsilon_0 \underset{\sim}{E} \, \nabla \cdot \underset{\sim}{E} + \frac{1}{\mu_0} (\nabla \times \underset{\sim}{B}) \times \underset{\sim}{B}$$

$$- \epsilon_0 \; \underset{\sim}{E} \times (\nabla \times \underset{\sim}{E}) \Big]$$

$$\underset{\sim}{E} \times (\nabla \times \underset{\sim}{E}) = \frac{1}{2} \nabla |\underset{\sim}{E}|^2 - \underset{\sim}{E} \cdot \nabla \underset{\sim}{E}$$

$$\underset{\sim}{B} \times (\nabla \times \underset{\sim}{B}) = \frac{1}{2} \nabla |\underset{\sim}{B}|^2 - \underset{\sim}{B} \cdot \nabla \underset{\sim}{B}$$

$$\frac{d}{dt}\left(\underset{\sim}{P}_{mech} + \underset{\sim}{P}_{em}\right) = \int d\underset{\sim}{x} \left[\epsilon_0 \left(\underset{\sim}{E} \nabla \cdot \underset{\sim}{E} - \frac{1}{2} \nabla E^2 + \underset{\sim}{E} \cdot \nabla \underset{\sim}{E} \right) + \frac{1}{\mu_0} \left(\underset{\sim}{B} \cdot \nabla \underset{\sim}{B} - \frac{1}{2} \nabla B^2 \right) \right]$$

$$= \int d\underset{\sim}{x} \left[\epsilon_0 \left(\nabla \cdot \underset{\sim}{E}\,\underset{\sim}{E} - \frac{1}{2} \nabla E^2 \right) + \frac{1}{\mu_0} \left(\nabla \cdot \underset{\sim}{B}\,\underset{\sim}{B} - \frac{1}{2} \nabla B^2 \right) \right.$$

The RHS can be written as the divergence of the Maxwell stress tensor

$$\underset{\approx}{T} = \epsilon_0 \underset{\sim}{E}\,\underset{\sim}{E} + \frac{1}{\mu_0} \underset{\sim}{B}\,\underset{\sim}{B} - \frac{1}{2} \underset{\approx}{I} \left(E^2 + B^2 \right)$$

where $\underset{\approx}{I}$ is the unit tensor

$$\frac{d}{dt}\left(\underset{\sim}{P}_{mech} + \underset{\sim}{P}_{em}\right) = \int d\underset{\sim}{x}\ \nabla \cdot \underset{\approx}{T}$$

$$= \int_S d\underset{\sim}{S} \cdot \underset{\approx}{T}$$

$\underset{\approx}{T} \cdot \hat{n}$ is the force per unit area acting on the enclosed volume with $\hat{n}$ the normal to the surface.