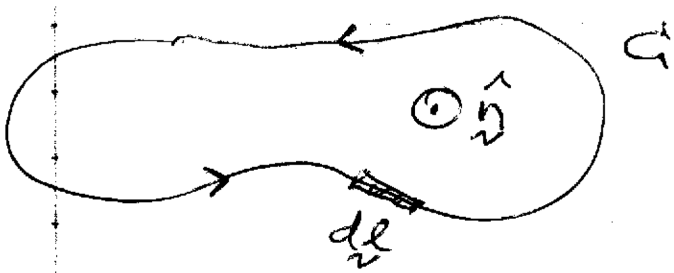
Faraday's law of induction

Up to this point we have considered magnetic and electric fields produced by steady state currents and charge distributions. We now want to begin to generalize these results to allow for time variation.

Consider a circuit C as shown:



The total magnetic flux linking the circuit is

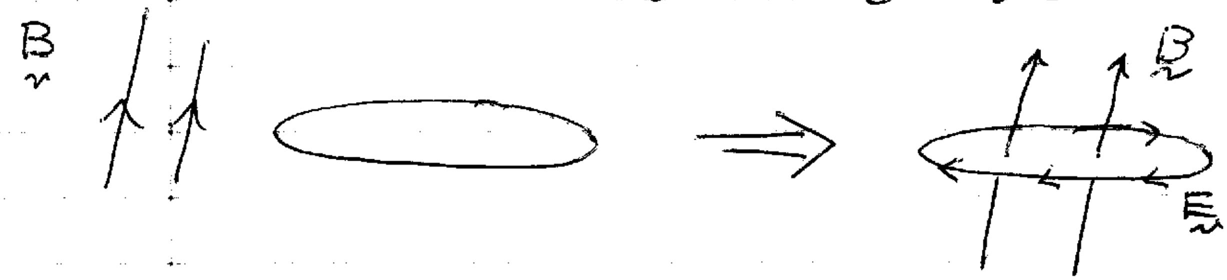
$$F = \int_S d\underset{\sim}{S} \cdot \underset{\sim}{B}$$

with $d\underset{\sim}{S} = \hat{\underset{\sim}{n}}\, dS$ given by the right hand rule. Observations are that an electric field is produced in this circuit if the magnetic flux linking the circuit changes in time. Define

$$\mathcal{E} = \oint \underset{\sim}{E} \cdot d\underset{\sim}{l} = \text{electromotive force}$$

$$\mathcal{E} = -\frac{d}{dt} F \qquad \text{Faraday's law}$$

The direction of the electric field is such that the resulting current would oppose the change in the magnetic field
$\Rightarrow$ Lenz' law



The result applies to an arbitrary closed loop in space $\Rightarrow$ no conductor needed

$$\mathcal{E} = \oint_C \mathbf{E}\cdot d\mathbf{l} = \int_S \nabla\times\mathbf{E}\cdot d\mathbf{S}$$

$$\Rightarrow \int d\mathbf{S}\cdot\left[\nabla\times\mathbf{E} + \frac{\partial}{\partial t}\mathbf{B}\right] = 0$$

Since this is valid for any surface S, the result must be valid locally at every point in space.

$$\nabla\times\mathbf{E} + \frac{\partial}{\partial t}\mathbf{B} = 0$$

Differential form of ~~Fara~~ Faraday's law

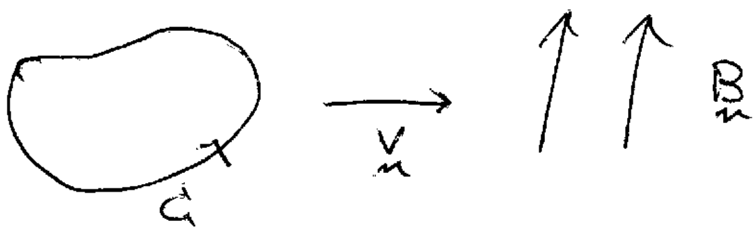
Note that for $\partial\mathbf{B}/\partial t = 0$ this reduces to

$$\nabla\times\mathbf{E} = 0$$

$\Rightarrow$ previous result for electrostatics

Galilean transformation of the electric field

Consider a magnetic field $\underset{\sim}{B}(\underset{\sim}{x},t)$ in the lab frame and a closed loop C moving with a uniform velocity $\underset{\sim}{V}$.



Let $\underset{\sim}{E}'$ be the electric field in the frame of the loop.

$$\oint \underset{\sim}{E}'\cdot d\underset{\sim}{l} = -\frac{d}{dt}\int_S \underset{\sim}{B}\cdot d\underset{\sim}{S}$$

where $\quad \frac{d}{dt} = \frac{\partial}{\partial t} + \underset{\sim}{V}\cdot\nabla$

$$\oint \underset{\sim}{E}'\cdot d\underset{\sim}{l} = -\int_S \frac{\partial \underset{\sim}{B}}{\partial t}\cdot d\underset{\sim}{S} - \int(\underset{\sim}{V}\cdot\nabla\underset{\sim}{B})\cdot d\underset{\sim}{S}$$

$$\nabla\times(\underset{\sim}{B}\times\underset{\sim}{V}] = \underset{\sim}{V}\cdot\nabla\underset{\sim}{B} - \underset{\sim}{V}\,\cancelto{0}{\nabla\cdot\underset{\sim}{B}}$$

where ∇ acts only on $\underset{\sim}{B}$ since $\underset{\sim}{v}$ is uniform.

$$\oint \underset{\sim}{E}'\cdot d\underset{\sim}{l} = -\int_S \frac{\partial \underset{\sim}{B}}{\partial t}\cdot d\underset{\sim}{S} - \underbrace{\int_S \left[\nabla\times(\underset{\sim}{B}\times\underset{\sim}{V})\right]\cdot d\underset{\sim}{S}}_{\oint \underset{\sim}{B}\times\underset{\sim}{V}\cdot d\underset{\sim}{l}}$$

$$\oint (\underset{\sim}{E}' - \underset{\sim}{V} \times \underset{\sim}{B}) \cdot d\underset{\sim}{l} = - \int_{S} d\underset{\sim}{S} \cdot \frac{\partial}{\partial t} \underset{\sim}{B}$$

But from Faraday's law in the lab frame

$$\frac{\partial}{\partial t} \underset{\sim}{B} + \nabla \times \underset{\sim}{E} = 0$$

or

$$\oint \underset{\sim}{E} \cdot d\underset{\sim}{l} = - \int d\underset{\sim}{S} \cdot \frac{\partial}{\partial t} \underset{\sim}{B}$$

where $\underset{\sim}{E}$ is the lab frame $\underset{\sim}{E}$. Thus

$$\underset{\sim}{E}' = \underset{\sim}{E} + \underset{\sim}{V} \times \underset{\sim}{B}$$

Galilean transform of $\underset{\sim}{E}$ valid for $|\underset{\sim}{V}| \ll c$. In this limit $\underset{\sim}{B}$ is unchanged.

$\Rightarrow$ In Faraday's law must be careful that in

$$\oint \underset{\sim}{E}' \cdot d\underset{\sim}{l} = - \frac{d}{dt} \int_{S'} \underset{\sim}{B} \cdot d\underset{\sim}{S}'$$

$\underset{\sim}{E}'$ is evaluated in the rest frame of the loop and the d/dt is the total time derivative.

$\Rightarrow$ any of the following can change the flux linking the loop.

$\Rightarrow$ $\dfrac{\partial \underset{\sim}{B}}{\partial t}$ time variation

$\Rightarrow$ $\underset{\sim}{V} \cdot \nabla \underset{\sim}{B}$ convection of $\underset{\sim}{B}$

$\Rightarrow$ change in loop structure

Equivalence with force on a moving charge

Consider a charge q moving with velocity $\underset{\sim}{v}$ in an electric field $\underset{\sim}{E}$, $\underset{\sim}{B}$ in the lab frame

$$\underset{\sim}{F} = q(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B})$$

In a frame moving with the charge, $\underset{\sim}{v} = 0$ and the force is

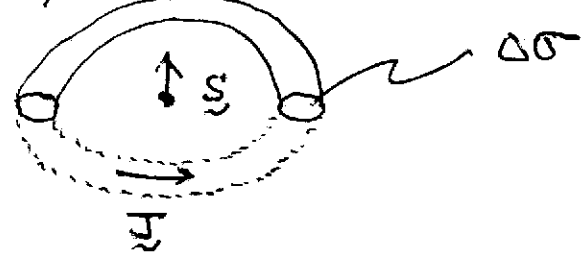
$$\underset{\sim}{F} = q\,\underset{\sim}{E}'$$

with $\underset{\sim}{E}'$ the electric field in the moving frame. The force is independent of frame for $|\underset{\sim}{v}| \ll c$ so

$$\underset{\sim}{E}' = \underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B} \quad \Rightarrow \text{as before}$$

Energy stored in magnetic fields

Want to consider the gradual buildup of a set of currents to a final state ~~with a~~ to determine the stored magnetic energy. Consider a loop of current



Change the magnetic field by $\delta \underset{\sim}{B}$.
The current loop will have a change in flux δF

$$\delta F = \int_S \delta \underset{\sim}{B} \cdot d\underset{\sim}{S}$$

This induces an emf $\delta \mathcal{E}$,

$$\delta \mathcal{E} = -\frac{\delta F}{\delta t} = \oint d\underset{\sim}{l} \cdot \delta \underset{\sim}{E}$$

This emf does work on an electron with velocity $\underset{\sim}{v}$ at a rate

$$\text{Force} \cdot \underset{\sim}{v} = -e\, \delta E \cdot \underset{\sim}{v}$$

On electrons with number density n,

$$\frac{\text{rate of work}}{\text{vol}} = -ne\, \underset{\sim}{v} \cdot \delta \underset{\sim}{E} = \underset{\sim}{J} \cdot \delta \underset{\sim}{E}$$

Integrating over the volume of the loop

$$\frac{\delta W_{loop}}{\delta t} = \oint \underset{\sim}{J} \cdot \delta \underset{\sim}{E}\, \Delta\sigma\, dl$$

$$= J \Delta\sigma \oint \delta E \cdot d\underset{\sim}{l}$$

$$= -J \Delta\sigma \frac{\delta F}{\delta t}$$

$$\Rightarrow \quad \delta W_{loop} = -J \Delta\sigma\, \delta F$$

$$= -J \Delta\sigma \int_S \delta \underset{\sim}{B} \cdot d\underset{\sim}{S}$$

To maintain $\underset{\sim}{J}$ as magnetic flux is added, external sources must supply this energy

$$\delta W = J \Delta\sigma \int_{S'} \delta \underset{\sim}{B} \cdot d\underset{\sim}{S}'$$

This is the change in energy of the current loop as magnetic flux is added. Want to express the result in terms of $\underset{\sim}{B}$ alone (not J). Can write

$$\int_{S'} \delta B \cdot d\underset{\sim}{S} = \int_{S'} \nabla \times \delta \underset{\sim}{A} \cdot d\underset{\sim}{S} = \oint \delta \underset{\sim}{A} \cdot d\underset{\sim}{l}$$

so

$$\delta W = J \Delta\sigma \oint \delta \underset{\sim}{A} \cdot d\underset{\sim}{l}$$

$$= \int d\underset{\sim}{x}\ \underset{\sim}{J} \cdot \delta \underset{\sim}{A}$$

since $d\underset{\sim}{x} = dl\, \Delta\sigma$ and $J\, d\underset{\sim}{l} = dl\, \underset{\sim}{J}$. The free current $\underset{\sim}{J}$ is given by

$$\underset{\sim}{J} = \nabla \times \underset{\sim}{H}$$

so

$$\delta W = \int d\underset{\sim}{x}\ \nabla \times \underset{\sim}{H} \cdot \delta \underset{\sim}{A} .$$

$$\nabla \cdot (\underset{\sim}{H} \times \delta \underset{\sim}{A}) = \delta \underset{\sim}{A} \cdot \nabla \times \underset{\sim}{H} - \underset{\sim}{H} \cdot \underbrace{\nabla \times \delta \underset{\sim}{A}}_{\delta \underset{\sim}{B}}$$

The divergence term goes away on integration over volume so

$$\delta W = \int d\mathbf{x}\ \mathbf{H} \cdot \delta \mathbf{B}$$

For $\mathbf{B} = \mu \mathbf{H}$,

$$\delta W = \int d\mathbf{x}\ \frac{1}{\mu}\ \mathbf{B} \cdot \delta \mathbf{B}$$

$$= \frac{1}{2} \int d\mathbf{x}\ \delta\left(\frac{1}{\mu}\ \mathbf{B} \cdot \mathbf{B}\right)$$

$$W = \frac{1}{2} \int d\mathbf{x}\ \frac{1}{\mu}\ B^2$$

$\Rightarrow$ energy density in a magnetic field is

$$w_B = \frac{B^2}{2\mu}$$

Energy associated with permeable material ~~added~~ moved into an external field $\mathbf{B}_0$,

$$\bar{W}_{\mathbf{M}} = -\frac{1}{2} \int d\mathbf{x}\ \mathbf{M} \cdot \mathbf{B}_0$$

Add a fixed magnetic dipole into B_0,

$$W_M = -\mathbf{m} \cdot \mathbf{B}_0 \quad \text{with } \mathbf{m} = \int d\mathbf{x}\ \mathbf{M}$$

$\Rightarrow$ no $1/2$