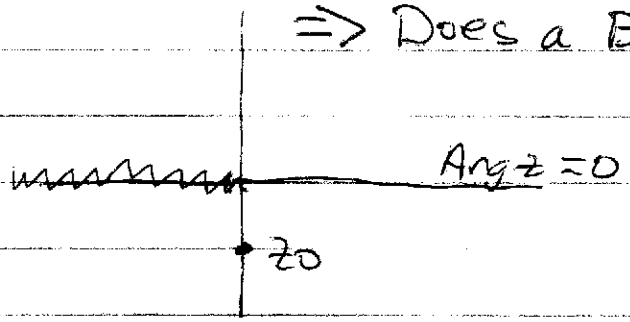
Overview of complex variables

$w(z) = f(z)$ defines a mapping from the complex z plane to the complex w plane

⇒ For functions such as $\ln(z)$ or any fractional power require a branch cut to force the values to be single valued.

⇒ Does a BC have to extend to infinity?



What is Arg z just below the cut? Above the cut?

What is $\mathrm{Arg}\left[(z_0)^{1/2}\right]$?

For the derivative of $f(z)$ to exist, the derivative must be the same when taken in any direction in the z plane. This requires that the Cauchy-Riemann conditions be satisfied.

For $f = u + iv$

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y} \end{aligned}\right\} \text{CR condition}$$

A function is analytic ~~if it is~~ at z_0 if the derivative exists at z_0 and everywhere in a neighborhood.

Cauchy Goursat Theorem

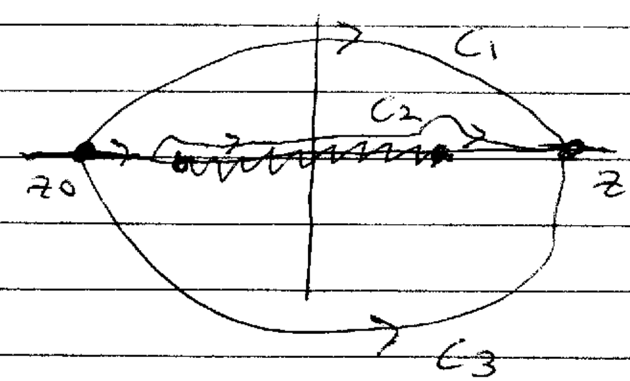
If a function is analytic in a simply connected region R, then

$$\oint_C dz\, f(z) = 0$$

Path independence of integrals for f analytic

$$\int_{z_0, C_1}^{z_1} dz\, f(z) = \int_{z_0, C_2}^{z_1} dz\, f(z)$$

Can choose any C_1 or C_2 as long as don't cross an analytic region in moving from C_1 to C_2



C_1, C_2 yield same values

C_3 will yield a different value.

Cauchy's integral formula

$$f(z_0) = \frac{1}{2\pi i} \oint_C dz\, \frac{f(z)}{z - z_0}$$ for f analytic within C.

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C dz\, \frac{f(z)}{(z - z_0)^{n+1}}$$

Singularities and Poles

If $f(z)$ is ~~singular~~ singular at z_0 but analytic everywhere in a neighborhood then z_0 is an isolated singular point

$$f(z) = \frac{1}{z^{1/2}(z-1)^2}$$

$f(z)$ has an isolated sing. point at $z=1$

$\Rightarrow$ second order pole

The singularity at $z=0$ is not an isolated sing. pt.

Taylor series

A function can be expanded in a Taylor series around a point z_0 where f is analytic.

The series converges at a radius given by the distance to the nearest singularity

$$f(z) = \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{n!} f^{(n)}(z_0)$$

Laurent Series

A function $f(z)$ can be expanded in a Laurent series around an isolated singular point

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

$$a_n = \frac{1}{2\pi i} \oint_C dz' \frac{f(z')}{(z'-z_0)^{n+1}}$$

$\Rightarrow$ usually better not to calculate a_n from integral.

__example__ $f(z) = \dfrac{1}{(z-3)^3 z}$

Expand in a Laurent series around $z=3$. Since z^{-1} is analytic at $z=3$, expand z^{-1} in a Taylor series around $z=3$ and multiply by $(z-3)^{-3}$

$$f(z) = \frac{1}{(z-3)^3}\frac{1}{3}\frac{1}{\left(1+\frac{z-3}{3}\right)}$$

$$= \frac{1}{3}\frac{1}{(z-3)^3}\sum_{n=0}^{\infty}\left(\frac{z-3}{3}\right)^{n}(-1)^n$$

__Residue Theorem__

$$\oint_C dz\, f(z) = 2\pi i \sum_j a_{-1}(z_j)$$

$= 2\pi i$ sum of residues inside of C

__Evaluating residues__

$\Rightarrow$ for mth order pole at z_j

$$a_{-1}(z_j) = \frac{1}{(m-1)!}\left[(z-z_j)^m f(z)\right]^{(m-1)}\Big|_{z=z_j}$$

$\Rightarrow$ first order pole

$f(z) = \dfrac{P(z)}{Q(z)}$ with $Q(z_j) = 0$

$$a_{-1} = \frac{P(z_j)}{Q'(z_j)}$$

Evaluating integrals

① Be sure that the contour does not cross a singular point $\Rightarrow$ need to define contour with respect to sing. pt.

$\Rightarrow$ exceptions: some singularities are integrable, e.g. $\ln(z)$, $z^{-1/2}$ - - - -

② For integrals extending to infinity, close the contour at ∞. Make sure the contribution from the contour at ∞ is zero.

$\Rightarrow$ for f going to zero faster than z^{-2} can close

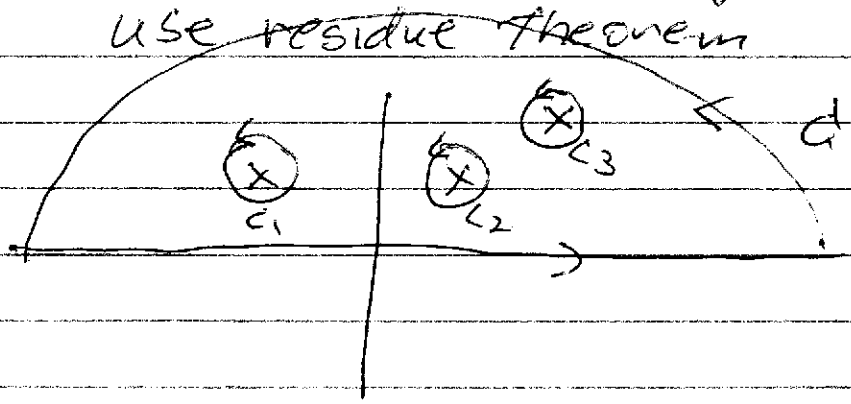
Jordan's Lemma

$\Rightarrow$ for $f \sim e^{iz} g(z)$ close in UHP as long as $g \to 0$ as $z \to \infty$

$\Rightarrow$ for $f \sim e^{-iz} g(z)$ close in LHP.

since $e^{-iz} = e^{-i(x+iy)} = e^{-ix} e^{y}$
$\to 0$ in LHP

③ Once have closed integral shrink contour around enclosed singularities and use residue theorem



Saddle Point Techniques

Can usually evaluate integrals when there's a large parameter

$\Rightarrow$ most special functions in physics have integral representations

$\Rightarrow$ can evaluate for large argument e.g. $J_\nu(k\rho)$, $I_\nu(k\rho)$ ....

1. Identify the dominant terms in the integrand that control the topography of the integral
   $\Rightarrow$ be careful of branch cuts

2. Write the integral in the exponential form

$$I = \int_{c_i} dz\, e^{-h(z)}\, g(z)$$

where $h(z)$ has a large parameter and $g(z)$ only weakly impacts the integrand.

3. Locate the saddle points where $h'(z) = 0$

$$h'(z_{sp}) = 0 \Rightarrow z_{sp}$$

4. Deform c_i through to highest S.P. if the contour path crosses it.
   $\Rightarrow$ the contour might not cross all saddle points

5 Expand $h(z)$ around z_{sp}

$$h(z) \simeq h(z_{sp}) + \tfrac{1}{2} h''(z_{sp}) (z - z_{sp})^2$$

6 ~~Find the path of steepest descent where the phase of $h(z)$ does not change.~~

$h''(z_{sp})(z - z_{sp})^2$ is real, and positive

$z - z_{sp} \equiv s e^{i\theta}$ $\Rightarrow$ choose θ to find the PSD

$$I = e^{-h(z_{sp})} g(z_{sp}) \int_{-\infty}^{\infty} ds\, e^{i\theta} e^{-\frac{1}{2} h''(z_{sp}) s^2 e^{2i\theta}}$$

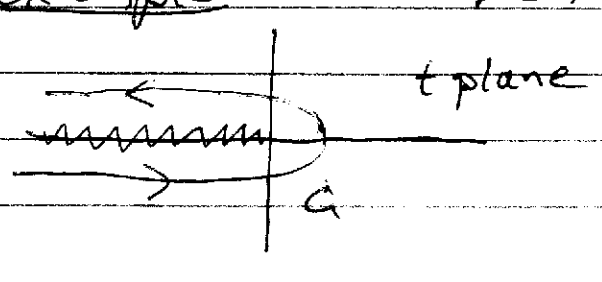
$$= e^{-h(z_{sp})} g(z_{sp}) e^{i\theta} \sqrt{\pi} \left(\frac{2}{h'' e^{2i\theta}}\right)^{1/2}$$

where extend integral over s to $\pm\infty$ since e^{-h} quickly goes to zero away from S.P.

Analytic Continuation of Contour Integrals

Contour integrals of functions typicall remain defined only for a range of phase angles

example $\quad Q_\nu(z) = \int_C dt\, t^\nu e^{zt}$



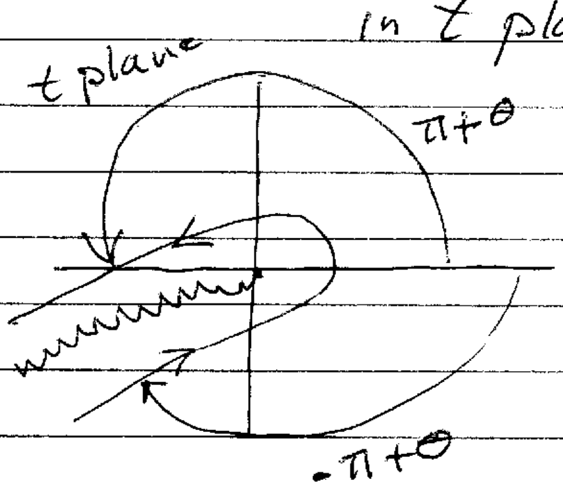
$\Rightarrow$ integral defined for $Re(z) > 0$

$\Rightarrow$ so $e^{zt} \to 0$ as $t \to -\infty$

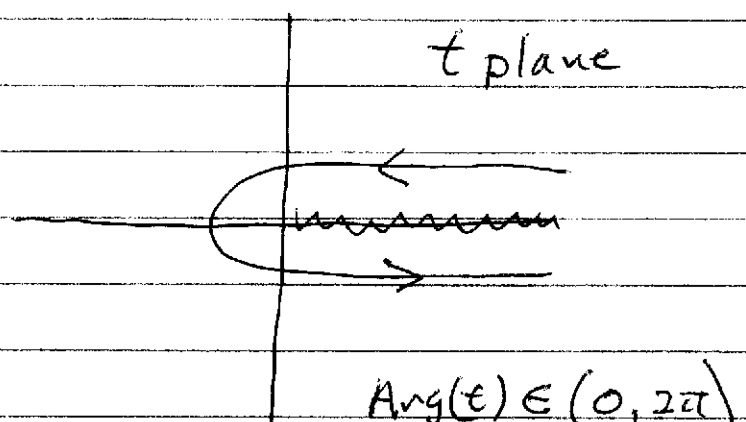
How can we evaluate $Q_\nu(re^{-i\pi})$?

$$Q_\nu(re^{-i\theta}) = \int_C dt\, t^\nu e^{re^{-i\theta}t}$$

$\Rightarrow$ gradually increase θ from 0 to π while deforming the integal and BC in t plane so the $e^{-i\theta}$ is cancelled.



$\Rightarrow$



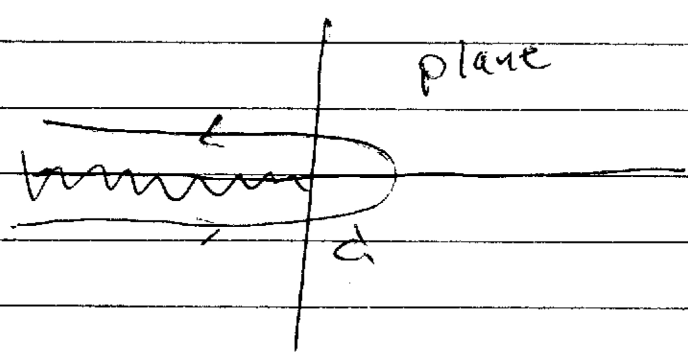
$\mathrm{Arg}(t) \in (0, 2\pi)$

Now change variables $t = pe^{i\pi}$ $\quad dt = dp\,e^{i\pi}$

$$p = te^{-i\pi}$$

$$Q_\nu(re^{-i\pi}) = e^{i\pi\nu} e^{i\pi} \int dp\, p^\nu e^{pr} \quad \Rightarrow \arg(p) \in (-\pi, \pi)$$

as in original integral



$$Q_\nu(re^{i\pi}) = -e^{i\pi\nu} Q_\nu(r)$$