Various other techniques

Reduction of order:

Suppose have found a solution $\varphi_1(x)$ of the equation $\mathcal{L}(y)=0$, then let

$$y(x) = u(x)\,\varphi_1(x)$$

with

$$\mathcal{L}(u\varphi_1) = 0$$

This yields an nth order equation for u

$$\Rightarrow \mathcal{M}(u) = 0$$

but this equation has no term proportional to $u^{(0)}$. Thus, $\mathcal{M}(u)=0$ is an $(n-1)$th order equation for $u' = du/dx$.

<u>example</u> Consider the general second order equ

$$y'' + a_1(x)\,y' + a_0(x)\,y = 0$$

with a solution $\varphi_1(x)$. Let $y = u(x)\,\varphi_1(x)$

$$y' = u'\varphi_1 + u\varphi_1'$$
$$y'' = u''\varphi_1 + 2u'\varphi_1' + u\varphi_1''$$

The equation becomes

$$u''\varphi_1 + 2u'\varphi_1' + \underline{u\varphi_1''} + a_1(u'\varphi_1 + \underline{u\varphi_1'}) + \underline{a_0 u\varphi_1} = 0$$

The underlined terms cancel since $\mathcal{L}(\varphi_1) = 0$
so

$$u''\varphi_1 + 2u'\varphi_1' + a_1 u'\varphi_1 = 0$$

$$\frac{d}{dx}(u'\varphi_1^2) + a_1(u'\varphi_1^2) = 0$$

$\Rightarrow$ integrate this

$$u'\varphi_1 = c\, e^{-\int_{x_0}^{x} dx'\, a_1(x')}$$

$\Rightarrow$ integrate to obtain $u(x)$ and $\varphi_2 = u\varphi_1$

$$\varphi_2 = \varphi_1(x) \int_{x_0}^{x} dx' \, \frac{e^{-\int_{x_0}^{x'} dx''\, a_1(x'')}}{\varphi_1^2(x')}$$

Exact Equations:

An exact equation is the derivative of an equation of lower order

$$\mathcal{L}(y) = \frac{d}{dx}\mathcal{M}(y) = 0$$

$$\mathcal{M}(y) = c = \text{const.}$$

$\Rightarrow$ inhomogeneous eqn of lower order

$\Rightarrow$ discuss inhomogeneous eqns later

<u>example</u> $y'' + xy' + y = 0$

Can be re-written as

$$\frac{d}{dx}(y' + xy) = 0$$

or $$y' + xy = c$$

$\Rightarrow$ readily soluble

Can sometimes multiply an equation by an integrating factor to make it exact

example

$$y'' + \frac{1+x}{x} y' + \frac{x-1}{x^2} y = 0$$

$\Rightarrow$ multiply by e^x

$$e^x y'' + \frac{1+x}{x} e^x y' + \frac{x-1}{x^2} e^x y$$

$$= \frac{d}{dx}\left(e^x y' + \frac{e^x}{x} y\right) = 0$$

$$y' + \frac{1}{x} y = ce^{-x}$$

$\Rightarrow$ readily soluble