Classification of singular points

1) <u>Ordinary point</u> An ordinary point x_0 of a differential equation is a point where all of the a_j's are analytic.

$\Rightarrow$ must be in standard form

$$y^{(n)} + a_{n-1}(x)\, y^{(n-1)} + \cdots$$

2) <u>Regular singular point</u> The point x_0 is a regular singular point if the equation can be written in the form

$$(x-x_0)^n y^{(n)}(x) + (x-x_0)^{n-1} \hat{a}_{n-1}(x)\, y^{(n-1)}(x) + \cdots (x-x_0)^0 \hat{a}_0(x)\, y^{(0)}(x) = 0$$

where all of the $\hat{a}_{n-1}, \ldots \hat{a}_0$ are analytic at x_0.

$\Rightarrow$ established method to obtain solution

3) <u>Irregular singular points</u> The point x_0 is an irregular singular point if it is neither an OP or a RSP.

$\Rightarrow$ no established method of solution

Classification of the point at $x = \infty$

Can map the point at ∞ to the origin by using the inversion transformation

$$x = \frac{1}{t}$$

$$\frac{d}{dx} = \frac{dt}{dx}\frac{d}{dt}\ , \quad \frac{dt}{dx} = -\frac{1}{x^2} = -t^2$$

$$\frac{d}{dx} = -t^2\frac{d}{dt}$$

$$\frac{d^2}{dx^2} = \left(-t^2\frac{d}{dt}\right)\left(-t^2\frac{d}{dt}\right) = t^4\frac{d^2}{dt^2} + 2t^3\frac{d}{dt}$$

Continue and classify $t=0$ as ~~as~~ an ordinary point, a RSP or an ISP.

<u>example</u> $\quad y'' = e^x y$

Every point is ordinary except ∞.

$$t^4\frac{d^2}{dt^2}y + 2t^3\frac{d}{dt}y = e^{1/t}y$$

$$t^2 y_{tt} + 2t y_t = \underbrace{\frac{1}{t^2}e^{1/t}}_{\Rightarrow \infty \text{ at } t=0} y$$

Irregular sing. pt. at $x = \infty$.

<u>example</u> $x^5 y''' = y$

$\Rightarrow x^3 y''' - \frac{1}{x^2} y = 0$

Irregular sing. point at $x=0$.

At infinity

$\Rightarrow t^3 y_{ttt} + 6t^2 y_{tt} + 6t y_t + t^2 y = 0$

$\Rightarrow$ RSP at $x = \infty$.

<u>example</u> $y' - |x| y = 0$

$\Rightarrow$ no ordinary points since $|x|$ is not analytic

<u>example</u> $x^2 y'' + x y' - y = 0$

$\Rightarrow$ RSP at $x=0$.

Euler equation

Consider the following equation

$$f(y) \equiv x^n y^{(n)} + x^{n-1} \hat{a}_{n-1} y^{(n-1)} + \cdots x^0 \hat{a}_0 y^{(0)} = 0$$

where the $\hat{a}_j$'s are constants. This is an Euler equation. It has a RSP at $x=0$.

This equation has powerlaw solutions.

$$y \sim x^r$$

$$\frac{dy}{dx} = r x^{r-1}$$

$$y^{(n)} = r(r-1) \cdots (r-n+1) x^{r-n}$$

$$x^n y^{(n)} = r(r-1) \cdots (r-n+1) x^r$$

$$f(x^r) = g(r) x^r$$

$$g(r) = r(r-1) \cdots (r-n+1) + r(r-1) \cdots (r-n+2) \hat{a}_{n-1} + \cdots \hat{a}_0 = 0$$

$\Rightarrow$ nth order polynomial for r.

$\Rightarrow$ n solutions for r.

$\Rightarrow$ n powerlaw solutions for y

$$x^{r_1}, x^{r_2}, \ldots, x^{r_n}$$

If have degenerate solutions

$$f(r) = (r - r_0)^m g(r)$$

$\Rightarrow$ differentiate diff. eqn. with respect to r

$$\frac{d}{dr} \mathcal{L}(x^r) = \mathcal{L}(\ln x \; x^r)$$

$$= \left[m(r-r_0)^{m-1} g(r) + (r-r_0)^m g'(r) \right] x^r$$

$$+ (r-r_0)^m g(r) \ln(x) \, x^r$$

Set $r = r_0$

$$\Rightarrow \mathcal{L}(\ln x \; x^{r_0}) = 0$$

Continue

$$x^{r_0}, \; x^{r_0} \ln(x), \; x^{r_0} (\ln x)^2 \cdots x^{r_0} (\ln x)^{m-1}$$

are all solutions

$\Rightarrow$ similar to degenerate exponential solutions for constant coeff. eqns.

example

$$4x^2y'' + y = 0 \Rightarrow \text{RSP at } x=0$$

$$y \sim x^r$$

$$[4r(r-1)+1]x^r = 0$$

$$r^2 - r + \frac{1}{4} = 0$$

$$r = \frac{1 \pm \sqrt{1-4\frac{1}{4}}}{2} = \frac{1}{2}$$

$y \sim x^{1/2} \Rightarrow$ degenerate case

second solution $x^{1/2}\ln(x)$

$\Rightarrow$ can substitute into eqn to check.

$$y = C_1 x^{1/2} + C_2 x^{\frac{1}{2}} \ln x$$

$\Rightarrow$ solution requires branch cut

Solutions of equations with regular singular points : local behavior

$$(x-x_0)^n y^{(n)} + (x-x_0)^{n-1} \hat{a}_{n-1}(x) + \cdots = 0$$

where $\hat{a}_j(x)$ are analytic around $x = x_0$.

$\Rightarrow$ the $\hat{a}_j$'s can be expanded in a Taylor series around x_0

$$\hat{a}_j(x) = \hat{a}_j(x_0) + \hat{a}_j'(x+x_0) + \cdots$$

Lowest order form of the equation around x_0

$$(x-x_0)^n y^{(n)} + (x-x_0)^{n-1} \hat{a}_{n-1}(x_0) y^{(n-1)} + \cdots = 0$$

$\Rightarrow$ Eulers eqn

$\Rightarrow$ very close to singularity have a power law solution

<u>example</u> $x^2 y'' + \alpha x y' + x y = 0$

RSP at $x = 0$, $\hat{a}_1 = \alpha \Rightarrow \hat{a}_1(0) = \alpha$

$\hat{a}_0 = x \Rightarrow \hat{a}_0(0) = 0$

Close to the singularity

$$x^2 y'' + \alpha x y' = 0$$

$$y \sim x^p \Rightarrow [p(p-1) + \alpha p]\, x^p = 0$$

$$p = 0, \quad p = 1-\alpha$$

$$y = c_1 + c_2 x^{1-\alpha}$$

If. $\alpha = \underline{1} \Rightarrow y = c_1 + c_2 \ln x$

<u>example</u> Legendre's eqn

Series solutions of Legendre's eqn around $x=0$ breaks. down at $x = \pm 1$. What are the solutions near $x = \pm 1$?

$$(1-x^2)\, y'' - 2xy' + \alpha(\alpha+1)\, y = 0$$

Let $t = x-1 \Rightarrow \frac{d}{dx} = \frac{d}{dt}$

$$\underbrace{[1-(t+1)^2]}_{1-t^2-1-2t \,=\, -t(t+2)}\, y_{tt} - 2(t+1)\, y_t + \alpha(\alpha+1)\, y = 0$$

near $t = 0$

$$-2t\, y_{tt} - 2 y_t + \alpha(\alpha+1)\, y = 0$$

$$t^2 y_{tt} + t\, y_t - \cancelto{0}{\frac{\alpha}{2}(\alpha+1)\, t}\, y = 0$$

$$\hat{a}_1(0) = 1 \,, \quad \hat{a}_0(0) = 0$$

$$t^2 y_{tt} + t\, y_t = 0 \quad \Rightarrow y \sim t^r$$

$$\big(r(r-1) + r\big)\, t^r = 0$$

$$\Rightarrow r = 0 \quad \Rightarrow \text{degenerate}$$

$$y_1 = 1 \,, \quad y_2 = \ln(t) = \ln(x-1)$$

Series Solutions for eqns with regular singular points

A differential eqn with reg. sing. pts has at least one solution of the form

$$\varphi(x) = x^r \sum_{k=0}^{\infty} c_k x^k$$

where x^r corresponds to the Euler solution close to the sing. pt.

__example__ $\quad x^2 y'' + \alpha x y' + x y = 0$

$\Rightarrow$ recall Euler solutions $1, x^{1-\alpha}$

Assume $\varphi(x)$ as above

$$\varphi' = \sum_{k=0}^{\infty} (k+r) c_k x^{k+r-1}$$

$$\varphi'' = \sum_{k=0}^{\infty} (k+r)(k+r-1) c_k x^{k+r-2}$$

$$\sum_{k=0}^{\infty} \left[(k+r)(k+r-1) c_k x^{k+r} + \alpha c_k (k+r) x^{k+r} + c_k x^{k+r+1} \right] = 0$$

$\Rightarrow$ each coefficient of x^{k+r} must vanish

From $k=0 \Rightarrow$

$$\left[r(r-1) + \alpha r \right] c_0 = 0$$

$$q(r) = r^2 + r(\alpha - 1) = 0$$

$q(r)$ = indicial polynomial

$$\Rightarrow r = 0, \; r = 1 - \alpha.$$

$\Rightarrow$ Euler solutions

Shift sum over k in last term down by 1

$$\sum_{k=1}^{\infty} \left[q(k+r)\, c_k + c_{k-1} \right] x^{k+r} = 0$$

$$c_k\, q(k+r) + c_{k-1} = 0$$

$$c_k = - \frac{c_{k-1}}{q(k+r)}$$

1) First solution $r = 0$

$$q(k+r) = (k+r)^2 + (k+r)(\alpha - 1)$$

$$= k(k + \alpha - 1)$$

$$c_k = - \frac{c_{k-1}}{k(k+\alpha-1)}$$

$$c_0 \neq 0$$

$$c_1 = - \frac{c_0}{\alpha}$$

$$c_2 = - \frac{c_1}{2(\alpha+1)} = \frac{c_0}{2\alpha(\alpha+1)}$$

$$\mathcal{C}_1 = \sum_{k=1}^{\infty} (-1)^k \frac{x^k}{k!\ \alpha(\alpha+1)\cdots(\alpha+k-1)} + 1$$

2) Second solution $r = 1-\alpha$

$$g(k+r) = (k+1-\alpha)(k+1-\alpha+\alpha-1)$$

$$= k(k+1-\alpha)$$

$$c_k = -\frac{c_{k-1}}{k(k+1-\alpha)} = \frac{c_{k-1}}{k(\alpha-k-1)}$$

$$c_1 = \frac{c_0}{\alpha-2}$$

$$c_2 = \frac{c_1}{2(\alpha-3)} = \frac{c_0}{2(\alpha-3)(\alpha-2)}$$

$$c_k = c_0 \frac{1}{k!\ (\alpha-2)(\alpha-3)\cdots(\alpha-k-1)}$$

$$\mathcal{C}_2 = x^{1-\alpha} \sum_{k=1}^{\infty} \frac{(-1)^k x^k}{k!\ (2-\alpha)(3-\alpha)\cdots(k+1-\alpha)}$$

$$+ x^{1-\alpha}$$

Do the series converge?

$$\mathcal{C} = \sum_{k=0}^{\infty} R_k \Longrightarrow \lim_{k\to\infty} \frac{R_{k+1}}{R_k} = \frac{x}{g(k+r)}$$

$$= \lim_{k\to\infty} \frac{x}{k^2} \longrightarrow 0$$

Converges for all x.

Degenerate roots

For $\alpha = 1$ the two series are identical.

$\Rightarrow$ second solution starts with

$\ln(x) \Rightarrow$ diverges at $x=0$

$\Rightarrow$ such a solution is typically discarded in physics problems where we are not interested in singular solutions

Other special cases

For roots separated by an integer

$\Rightarrow \alpha$ an integer

$\Rightarrow$ one of the series blows up.

$\Rightarrow$ series with most positive r ok.

$$x^0, \quad x^{1-\alpha}$$

$\Rightarrow$ most important for this class is to determine the behavior near $x=0$ to select a solution that does not diverge.

Frobenius Solutions

Solutions for second order equations with regular singular points

Consider:

$$x^2 y'' + a(x)\, x\, y' + b(x)\, y = 0$$

with a, b analytic around $x=0$.

Let r_1, r_2 be the roots of the indicial polynomial

$$q(r) = r(r-1) + a(0)\, r + b(0)$$

Assume $\mathrm{Re}(r_1) \geq \mathrm{Re}(r_2)$

(1) If $r_1 \neq r_2$ and $r_1 - r_2$ is not an integer, have two series solutions

$$\varphi_1 = x^{r_1} \sigma_1(x), \quad \varphi_2 = x^{r_2} \sigma_2(x)$$

where σ_1, σ_2 have series expansions which converge where a, b are analytic and $\sigma_1(0), \sigma_2(0) \neq 0$.

(2) If $r_1 = r_2$ have solutions

$$y_1 = x^{r_1}\sigma_1(x), \quad y_2 = \ln(x)\, y_1(x) + x^{r_1+1}\sigma_2(x)$$

with $\sigma_1(0), \sigma_2(0) \neq 0$.

(3) If $r_1 - r_2$ is an integer, have the solutions

$$y_1 = x^{r_1}\sigma_1(x)$$

$$y_2 = c\ln(x)\, y_1(x) + x^{r_2}\sigma_2(x)$$

where $\sigma_1(0), \sigma_2(0) \neq 0$ and the constant c may be zero.