## Linear homogeneous eqns with variable coefficients

Consider a nth order diff. eqn of the form

$$y^{(n)} + a_{n-1}(x)\, y^{(n-1)} + \cdots\ a_0(x)\, y^{(0)}(x) = 0$$

where the $a_j(x)$ are analytic over the interval of interest
$\Rightarrow$ no singularities or branch cuts

The basic properties that we've derived for eqns. with constant coefficients are again valid when the $a_j(x)$ vary with $x$ but are analytic.

1) An nth order diff. eqn has $n$ linearly independent solutions.

2) Given any $n$ linearly independent solutions, any solution can be written as

$$y = c_1\, \varphi_1(x) + \cdots\ c_n\, \varphi_n(x)$$

3) A set of solutions $\varphi_i(x)$ are linearly independent if and only if

$$W \neq 0$$

4) Solutions of initial value problems exist and are ~~ug~~ unique

<u>Wronskian</u>

The Wronskian still satisfies the diff eqn

$$\frac{dW}{dx} = -\, a_{n-1}(x)\, \bar{W}(x)$$

The solution of this equation is changed since $a_{n-1}$ depends on $x$.

$$\underbrace{\frac{1}{\bar{W}} \frac{d\bar{W}}{dx}}_{\frac{d}{dx} \ln(\bar{W})} = -\, a_{n-1}(x)$$

$$\underbrace{\int_{x_0}^{x'} dx' \frac{d}{dx'} \ln \bar{W}}_{\ln\left[\frac{\bar{W}(x)}{\bar{W}(x_0)}\right]} = -\int_{x_0}^{x} dx'\, a_{n-1}(x')\, dx'$$

$$W(x) = W(x_0)\, e^{-\int_{x_0}^{x} dx'\, a_{n-1}(x')}$$

Note that as long as $a_{n-1}(x)$ is non-singular

$W(x) \neq 0$ if $W(x_0) \neq 0$

Example Legendre's Eqn

$$(1-x^2)\, y'' - 2x y' + \alpha(\alpha+1)\, y = 0$$

$\Rightarrow$ standard form

$$y'' - \frac{2x}{1-x^2} y' + \frac{\alpha(\alpha+1)}{1-x^2} y = 0$$

Coefficients analytic for $x \neq \pm 1$

$$a_1 = -\frac{2x}{1-x^2}$$

$$W(x) = W(0)\, e^{-\int_0^x dx' \frac{-2x'}{1-x'^2}}$$

$$= W(0) \exp\left[ -\ln(1-x'^2) \Big|_0^x \right]$$

$$= W(0) \exp\left[ -\ln(1-x^2) \right]$$

$$= \frac{W(0)}{1-x^2}$$ $\Rightarrow$ non zero except at $x = \pm 1$

## Solving Equations with Variable Coefficients

The exponential solutions no longer form a basis for equations with variable coefficients.

$\Rightarrow$ alternate methods

### Series solutions

If the $a_j$'s are analytic in the neighborhood of a point $x_0$, they can be expanded in a Taylor series around $x_0$

$\Rightarrow$ might expect the solutions $\phi_j(x)$ could also be expanded in a power series.

1) Fuchs showed that all $n$ linearly independent solutions are analytic in the neighborhood of an ordinary point, where the $a_j$'s are analytic

2) Taylor series are valid at least up to the nearest singularity of the $a_j$'s in the complex plane

3) The location of a singularity of a solution must correspond to a singularity of at least one of the $a_j$'s.

$\Rightarrow$ the solution can not have singularities elsewhere.

<u>example</u>

$$(1+x^2)\,y' + 2xy = 0$$

$\Rightarrow$ standard form

$$y' + \frac{2x}{1+x^2}\,y = 0$$

$\Rightarrow$ has a Taylor series solution around $x=0$ valid for $|x| < \underline{1}$ since this is the distance to the nearest singularity at $x = \pm i$

<u>Example</u> Airy's Eqn.

$$y'' - xy = 0$$

This equation is important since it has the form of a "turning point", where a propagating wave becomes evanescent.

To see this, pretend that $x$ is not a variable and look for exponential solutions as if had constant coefficients

$$y \sim e^{ikx}$$

$$-k^2 y - xy = 0$$

$$\Rightarrow k^2 = -x$$

$$\Rightarrow k = \pm(-x)^{\frac{1}{2}}$$

For $x < 0$, the solutions are oscillatory, corresponding to waves

$$y \sim e^{\pm i |x|^{1/2} x}$$

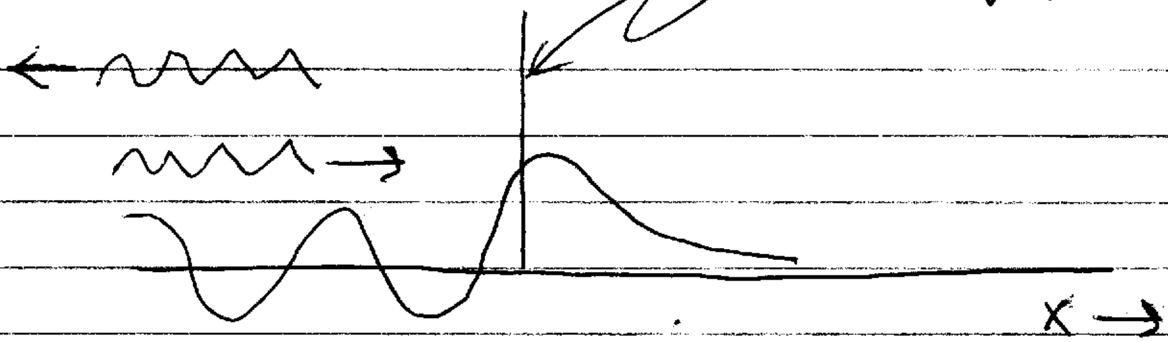
For $x > 0$, $k = \pm i x^{1/2}$

$$y \sim e^{\pm x^{3/2}}$$

$\Rightarrow$ exponentially increasing or decreasing

$\Rightarrow$ The point where $k \to 0$ is called a <u>turning point</u>.

$\Rightarrow$ a reflection point for waves turning point




Solve the Airy eqn using a power series.

$$\varphi = \sum_{k=0}^{\infty} c_k x^k$$

$$= c_0 + c_1 x + \cdots$$

$$\varphi' = \sum_{k=0}^{\infty} k c_k x^{k-1}$$

$$\varphi'' = \sum_{k=0}^{\infty} k(k-1) c_k x^{k-2}$$

Substituting into the Airy eqn,

$$\sum_{k=0}^{\infty} k(k-1) c_k x^{k-2} - \sum_{k=0}^{\infty} c_k x^{k+1} = 0$$

The equation must be satisfied for all x so the coefficient of each power of x must sum to zero.

Rewrite the first sum noting that $k=0$ and $k=1$ give zero

$$2c_2 + \underbrace{\sum_{k=3}^{\infty} k(k-1)c_k x^{k-2}}_{\text{shift } k \text{ by } 3} - \sum_{k=0}^{\infty} c_k x^{k+1} = 0$$

$$\sum_{k=0}^{\infty} (k+3)(k+2)c_{k+3} x^{k+1}$$

$$2c_2 + \sum_{k=0}^{\infty} \left[ (k+3)(k+2)c_{k+3} - c_k \right] x^{k+1} = 0$$

$$\Rightarrow c_2 = 0$$

$$\Rightarrow c_{k+3} = \frac{c_k}{(k+2)(k+3)}$$

Two series solutions: one with $c_0 \neq 0$ and one with $c_1 \neq 0$

<u>$c_0 \neq 0$</u>

$$c_3 = \frac{c_0}{2(3)}$$

$$c_6 = \frac{c_3}{5(6)} = \frac{c_0}{2(3)(5)(6)}$$

$$c_{3m} = \frac{c_0}{2\cdot 3\cdot 5\cdot 6 \cdots (3m-1)\cdot 3m}$$

$\underline{c_1 \neq 0}$

$$c_1 = \frac{c_4}{3 \cdot 4}$$

$$c_7 = \frac{c_4}{6 \cdot 7} = \frac{c_1}{3 \cdot 4 \cdot 6 \cdot 7}$$

$$c_{3m+1} = \frac{c_1}{3 \cdot 4 \cdot 6 \cdot 7 \cdots 3m \cdot (3m+1)}$$

$$c_5 = \frac{c_2}{4 \cdot 5} = 0$$

$$c_8 = 0$$

$$\vdots$$

$\Rightarrow$ Two series solutions for 2nd order eqn.

Let $$\varphi_0(x) = 1 + \sum_{m=1}^{\infty} \frac{x^{3m}}{2 \cdot 3 \cdot 5 \cdot 6 \cdots (3m-1)3m}$$

$$\varphi_1(x) = x + \sum_{m=1}^{\infty} \frac{x^{3m+1}}{3 \cdot 4 \cdot 6 \cdot 7 \cdots 3m \cdot (3m+1)}$$

General Solution

$$y = c_0 \varphi_0(x) + c_1 \varphi_1(x)$$

What about convergence?

Singularities of the coefficients?

Use the ratio test

$$\varphi_0 = 1 + \sum_{m=1}^{\infty} R_m$$

$$\lim_{m\to\infty} \frac{R_{m+1}}{R_m} = \frac{\dfrac{x^{3m+3}}{2\cdot 3\cdot \cdots (3m+2)(3m+3)}}{\dfrac{x^{3m}}{2\cdot 3 \cdots (3m-1)\cdot 3m}}$$

$$= \lim_{m\to\infty} \frac{x^3}{(3m+2)(3m+3)} = \lim_{m\to\infty} \frac{x^3}{9m^2}$$

$$\Rightarrow 0$$

Series converges for all $|x| < \infty$.

Linearly independent?

$\Rightarrow$ check that $W(x=0) \neq 0$

$\varphi_0(0) = 1$ , $\varphi_0'(0) = 0$

$\varphi_1(0) = 0$ , $\varphi_1'(0) = 1$

$W(0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \Rightarrow$ linearly indep.

Example Legendre's Eqn.

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$$

Already showed that

$$a_1 = \frac{-2x}{1-x^2}, \quad a_0 = \frac{\alpha(\alpha+1)}{1-x^2}$$

are analytic except at $x = \pm 1$.

$\Rightarrow$ look for series solution for $|x| < 1$

$$\varphi = \sum_{k=0}^{\infty} c_k x^k$$

$\Rightarrow$ note only positive powers of $x$ since $x=0$ is an ordinary point

$\Rightarrow$ $\varphi$ can not be singular at $x=0$.

$$\varphi' = \sum_{k=0}^{\infty} k c_k x^{k-1}$$

$$\varphi'' = \sum_{k=0}^{\infty} k(k-1) c_k x^{k-2}$$

$$\sum_{k=0}^{\infty} \Big[ \underbrace{k(k-1) c_k x^{k-2}}_{\text{shift } k \text{ up by } 2} - k(k-1) c_k x^k - 2k c_k x^k + \alpha(\alpha+1) c_k x^k \Big] = 0$$

$$\sum_{k=0}^{\infty} \Big[ (k+2)(k+1) c_{k+2} - [k(k-1) + 2k - \alpha(\alpha+1)] c_k \Big] x^k = 0$$

Coefficient of each power of $x$ must vanish

$$c_{k+2} = \frac{k(k+1) - \alpha(\alpha+1)}{(k+1)(k+2)} c_k$$

Two series solutions: one with $c_0 \neq 0$ and one with $c_1 \neq 0$

$$\varphi_0(x) = 1 - \frac{\alpha(\alpha+1)}{2!} x^2 + \frac{(\alpha-2)\alpha(\alpha+1)(\alpha+3)}{4!} x^4 + \cdots$$

$k \Rightarrow 2m$

$$= 1 + \sum_{m=1}^{\infty} (-1)^m \frac{(\alpha+2m-1)(\alpha+2m-3)\cdots(\alpha+1)\alpha(\alpha-2)\cdots(\alpha-2m+2)}{(2m)!}$$

$\otimes \; x^{2m} \Rightarrow$ even solution

$K \Rightarrow 2m+1$

$$\mathcal{Q}_1(x) = x + \sum_{m=1}^{\infty} (-1)^m \frac{(\alpha+2m)(\alpha+2m-2)\cdots(\alpha+2)(\alpha-1)(\alpha-3)\cdots(\alpha-2m+1)}{(2m+1)!} \, x^{2m+1}$$

So $y = c_0 \mathcal{Q}_0(x) + c_1 \mathcal{Q}_1(x)$

Do the series converge?

$$\lim_{m\to\infty} \frac{R_{m+1}}{R_m} = \lim_{m\to\infty} x^2 \frac{C_{m+2}}{C_m}$$

$$= \lim_{m\to\infty} x^2 \frac{m(m+1) - \alpha(\alpha+1)}{(m+1)(m+2)}$$

$$= \lim_{m\to\infty} x^2 = x^2$$

$\Rightarrow$ convergence $|x^2| < 1$

## Integer values of $\alpha$

One of the two series will truncate

$$C_{k+2} \propto k(k+1) - \alpha(\alpha+1) = 0$$

$$\Rightarrow k = \alpha, -(\alpha+1)$$

(1) For $\alpha$ an even integer,

$\mathcal{U}_0(x)$ truncates

$$C_k \neq 0 \quad \text{but} \quad C_{k+2} = 0$$

$$C_\alpha \neq 0 \qquad C_{\alpha+2} = 0$$

Highest surviving power is

$$C_\alpha x^\alpha$$

$\Rightarrow \mathcal{U}_0(x)$ is a polynomial of order $\alpha$. $\Rightarrow$ series valid for all $x$

$\Rightarrow \mathcal{U}_1(x)$ does not truncate so series requires $|x| < 1$

(2) For $\alpha$ an odd integer

$\Rightarrow$ odd series truncates for $k = \alpha$

$\Rightarrow$ $a_1(x)$ is a polynomial of order $\alpha$

$\Rightarrow$ $a_0(x)$ does not truncate

$\Rightarrow$ Polynomial solutions for

$\alpha = 0, 1, 2, 3, \ldots$

are <u>Legendre polynomials</u>