Linear Differential Equations with Constant Coefficients Arfken Chapter 7

Consider an nth order equation of the following form

$$y^{(n)} + a_{n-1} y^{(n-1)} + \cdots a_0 y^{(0)} = b(x)$$

$$\Rightarrow \mathcal{L}(y) = b(x)$$

with $y^{(j)}$ the jth derivative of y with respect to x. Take $a_0, a_1, \ldots a_{n-1}$ indep. of x.

Consider first $b(x) = 0 \Rightarrow$ homogeneous eqn

Eqn. is linear since first order in y.

Consider some m solutions φ_j of the eqn. Then any combination

$$y = \sum_j c_j \varphi_j(x)$$

is a solution since

$$\mathcal{L}(y) = \sum_j c_j \mathcal{L}(\varphi_j) = 0$$

Exponential solutions

Look for solutions of the form $y = e^{kx}$

$$\frac{d^n}{dx^n} y = k^n e^{kx} = k^n y$$

$$\mathcal{L}(y) = 0 \Rightarrow P(k) = \sum_{j=0}^{n} a_j k^j = 0$$

This is an nth order polynomial which has n solutions. Generally have n solutions of $\mathcal{L}(y) = 0$ of the form

$$e^{k_1 x}, e^{k_2 x} \cdots\cdots, e^{k_n x}$$

What if have repeated roots?

$$P(k) = g(k)(k-k_0)^m$$

with $g(k_0) \neq 0$

$$\mathcal{L}(e^{kx}) = g(k)(k-k_0)^m e^{kx}$$

Take the derivative with respect to k

$$\frac{d}{dk}\mathcal{L}(e^{kx}) = \mathcal{L}(xe^{kx}) = m\,g(k)(k-k_0)^{m-1} + g'(k)(k-k_0)^m$$

Set $k = k_0$

$$\mathcal{L}(xe^{k_0 x}) = 0$$

so $xe^{K_0 x}$ is a solution. Repeat to find

$e^{K_0 x}, xe^{K_0 x} \ldots x^{m-1} e^{K_0 x}$ are solutions.

Linear Independence

Consider a set of n solutions to $\mathcal{L}$, $\varphi_j(x)$. These solutions are linearly dependent if a linear combination can be found such that

$$\sum_j c_j \varphi_j(x) = 0$$

over the interval x of interest with $c_j \neq 0$ for at least some values of j. If the only solution is $c_j = 0$ for all j, the solutions are linearly independent.

Proof that the exponential solutions are linearly independent.

$\Rightarrow$ assume no repeated roots

$\Rightarrow$ proof by contradiction

Assume $\sum_j c_j e^{K_j x} = 0$ with at least

one value $c_q \neq 0$

$$\sum_{j=1}^{n} c_j e^{k_j x} = 0$$

Divide by $e^{k_1 x}$

$$\sum_{j=1}^{n} c_j e^{(k_j - k_1)x} = 0$$

Take x derivative $\Rightarrow$ $j = 1$ term goes away

$$\sum_{j=2}^{n} c_j (k_j - k_1) e^{(k_j - k_1)x} = 0$$

Continue for all $j \neq q$, are ~~..~~ left with

$$c_q (k_q - k_1)(k_q - k_2)(k_q - k_{q-1})(k_q - k_{q+1})$$
$$\otimes \cdots (k_q - k_n) e^{k_q x} = 0$$

$\Rightarrow$ $c_q = 0$ $\Rightarrow$ contradicts assumption

$\Rightarrow$ only solution is for $c_j = 0$ for all j

$\Rightarrow$ the exponential solutions are linearly independent.

Initial Value Problems

Consider the operator $L(y)=0$. Suppose that $y(x)$ and its first $n-1$ derivatives are specified at x_0,

$$y(x_0) = \alpha_0$$
$$y^{(1)}(x_0) = \alpha_1$$
$$\vdots$$
$$y^{(n-1)}(x_0) = \alpha_{n-1}$$

The solution y is uniquely defined.

Uniqueness Theorem

Consider the initial value problem as defined above. There exists at most one solution satisfying these equations.

Proof: Suppose y_1 and y_2 are two solutions then $\psi = y_1 - y_2$ satisfies $L(\psi)=0$ with

$$\psi^{(i)}(x_0) = 0$$

for $i=0, 1, \ldots n-1$. All derivatives vanish at x_0. ~~Can expand~~ Calculate higher derivatives from the diff. eqn. Can expand $\psi(x)$ in a Taylor series around x_0. $\Rightarrow \psi(x) = 0$ around x_0

valid until the closest singularity.

Wronskian

The Wronskian $W(\varphi_1 \cdots \varphi_n)$ of n functions $\varphi_1 \cdots \varphi_n$ having $n-1$ derivatives on some interval is defined as

determinant

$$W \equiv \begin{vmatrix} \varphi_1 & \cdots & \varphi_n \\ \varphi_1' & \cdots & \varphi_n' \\ \vdots & & \\ \varphi_1^{(n-1)} & \cdots & \varphi_n^{(n-1)} \end{vmatrix}$$

Theorem If $\varphi_1 \cdots \varphi_n$ are n solutions of $\mathcal{L}(y)=0$ on an interval, they are linearly independent if and only if $W \neq 0$ for all x on the interval.

Proof: Suppose $W \neq 0$ for all x on interval.

$\Longrightarrow$ assume linearly dependent and find contradiction.

Let $c_1, c_2 \cdots c_n$ be constants such that

$$c_1 \varphi_1(x) + c_2 \varphi_2(x) + \cdots c_n \varphi_n(x) = 0$$

Then

$$c_1 \varphi_1' + \cdots c_n \varphi_n' = 0$$

$$c_1 \varphi_1'' +$$

$$\vdots$$

$$c_1 \varphi_1^{(n-1)} \cdots c_n \varphi_n^{(n-1)} = 0$$

what about

$$c_1 \varphi_1^{(n)} + \cdots c_n \varphi_n^{(n)} \stackrel{?}{=} 0$$

Is it an independent equation?

This is a set of linear homogeneous equations for $c_1, \cdots c_n$. The condition that they have a nonzero solution is that the determinant of the coefficients is zero. But this is the Wronskian which is assumed to be nonzero. The only solution is $c_j = 0$ for all j

$\Rightarrow$ If $W \neq 0$ then the $\varphi_j(x)$ are linearly independent.

Conversely, suppose $\varphi_1, \cdots \varphi_n$ are linearly independent solutions to $\mathcal{L}(y) = 0$.

$\Rightarrow$ Suppose $W(x_0) = 0$ and prove contradiction

If $w(x_0) = 0$,

$$c_1 \varphi_1(x_0) + \cdots c_n \varphi_n(x_0) = 0$$

$$\vdots$$

$$c_1 \varphi_1^{(n-1)}(x_0) \cdots + c_n \varphi_n^{(n-1)}(x_0) = 0$$

has a solution with some $c_j \neq 0$.

Consider $\psi \equiv c_1 \varphi_1(x) + \cdots c_n \varphi_n(x)$

$$\Longrightarrow \mathcal{L}(\psi) = 0$$

but $\psi(x_0) = 0$, $\psi'(x_0) = 0 \cdots$, $\psi^{(n-1)}(x_0) = 0$

$\Longrightarrow \psi(x)$ from earlier proof since Taylor series gives zero.

$\Longrightarrow \varphi_1, \varphi_2 \cdots \varphi_n$ not linearly independent

$\Longrightarrow$ contradiction so $w(x_0) \neq 0$.

Existence Theorem for Initial Value Problems

Let $\alpha_0, \alpha_1 \cdots\cdots \alpha_{n-1}$ be any real constants and x_0 any real number. Then there exists a solution $\mathcal{L}(y)=0$ on $-\infty < x < \infty$ satisfying

$$y(x_0) = \alpha_0$$
$$y'(x_0) = \alpha_1$$
$$\vdots$$
$$y^{(n-1)}(x_0) = \alpha_{n-1}$$

$\Rightarrow$ initial value problems always have solutions

$\Rightarrow$ All derivatives evaluated at a single x_0

Let $y(x) = c_1\varphi_1 + \cdots c_n\varphi_n(x)$ where the φ_j are linearly indep. Want to solve for the c_j's.

$$c_1\varphi_1(x_0) + \cdots c_n\varphi_n(x_0) = \alpha_0$$
$$\vdots$$
$$c_1\varphi_1^{(n-1)}(x_0) + \cdots\cdots = \alpha_{n-1}$$

The determinant of the matrix equation for the c_j's is the Wronskian which is non zero. Therefore, the c_j's have a unique solution $\Rightarrow y(x)$ satisfies the BC's.

Explicit Equation for $W(x)$

$$w = \begin{vmatrix} \phi_1 & \cdots & \phi_n \\ \vdots & & \\ \phi_1^{(n-1)} & \cdots & \phi_n^{(n-1)} \end{vmatrix}$$

$$\frac{dw}{dx} = \begin{vmatrix} \phi_1 & \cdots & \phi_n \\ \vdots & & \\ \phi_1^{(n-2)} & \cdots & \phi_n^{(n-2)} \\ \phi_1^{(n)} & \cdots & \phi_n^{(n)} \end{vmatrix}$$

Since derivatives on other rows would make two rows equal

$$\phi_1^{(n)} = -a_{n-1}\phi_1^{(n-1)} - \cdots a_0\phi_1^{(0)}$$

$\Rightarrow$ other rows can be subtracted to eliminate all but $\phi_1^{(n-1)}$

$$\frac{dw}{dx} = -a_{n-1}\begin{vmatrix} \phi_1 & \cdots & \phi_n \\ \vdots & & \\ \phi_1^{(n-1)} & \cdots & \phi_n^{(n-1)} \end{vmatrix} = -a_{n-1}w$$

$$\boxed{\frac{dw}{dx} = -a_{n-1}w}$$

$$w \sim e^{kx}$$

$$k e^{kx} = -a_{n-1} e^{kx}$$

$$\Rightarrow k = -a_{n-1}$$

$$w = A e^{-a_{n-1}x}$$

Suppose know $w(x_0)$

$$w(x) = w(x_0)\, e^{-a_{n-1}(x-x_0)}$$

$\Rightarrow$ If $w(x_0) \neq 0$ the $w(x) \neq 0$ anywhere (except ∞)

example

$$\frac{d^2y}{dx^2} + \alpha^2 y = 0 \Rightarrow y \sim e^{ikx}$$

$$(-k^2 + \alpha^2)\, y = 0 \, , \quad k = \pm\alpha$$

$$\varphi_1 = e^{i\alpha x} \qquad W = \begin{vmatrix} e^{i\alpha x} & e^{-i\alpha x} \\ i\alpha e^{i\alpha x} & -i\alpha e^{-i\alpha x} \end{vmatrix}$$

$$\varphi_2 = e^{-i\alpha x}$$

$$= -2i\alpha$$

$\Rightarrow W \neq 0$ for $\alpha \neq 0$

$\Rightarrow e^{\pm i\alpha x}$ are linearly indep.

Suppose $\alpha = 0$, $\varphi_1 = \varphi_2 = 1 \Rightarrow W = 0$

$\Rightarrow$ not linearly independent

$\Rightarrow$ Degenerate case

$\Rightarrow$ Other solution is x.

$$\varphi_1 = 1 \qquad W = \begin{vmatrix} 1 & x \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

$$\varphi_2 = x$$

φ_1, φ_2 linearly indep.

For $\alpha \neq 0$ can also choose

$$a_1 = \sin \alpha x \qquad W = \begin{vmatrix} \sin\alpha x & \cos\alpha x \\ \alpha\cos\alpha x & -\alpha\sin\alpha x \end{vmatrix}$$

$$a_2 = \cos\alpha x$$

$$= -\alpha(\sin^2\alpha x + \cos^2\alpha x)$$

$$= -\alpha \neq 0$$

$\Rightarrow$ linearly indep.

More than two solutions?

$\sin\alpha x,\ e^{i\alpha x},\ e^{-i\alpha x}$

$$W = \begin{vmatrix} \sin\alpha x & e^{i\alpha x} & e^{-i\alpha x} \\ \alpha\cos\alpha x & i\alpha e^{i\alpha x} & -i\alpha e^{-i\alpha x} \\ -\alpha^2\sin\alpha x & -\alpha^2 e^{i\alpha x} & -\alpha^2 e^{-i\alpha x} \end{vmatrix}$$

$$= -\alpha^3 \begin{vmatrix} \sin\alpha x & e^{i\alpha x} & e^{-i\alpha x} \\ \cos\alpha x & i e^{i\alpha x} & -i e^{-i\alpha x} \\ \sin\alpha x & e^{i\alpha x} & e^{-i\alpha x} \end{vmatrix} = 0$$

Top and bottom rows the same since 2nd derivative can always be expressed in terms of lower derivatives

⇒ This is why an nth order eqn has only n solutions

⇒ The Wronskian will always vanish since higher derivatives can always be expressed in ~~theorms~~ terms of lower derivatives

⇒ The bottom row will always be zero.

<u>Example</u> Initial Value Problem

$$\frac{d^2y}{dx^2} + \beta^2 y = 0 \qquad \begin{aligned} y(0) &= \alpha_0 \\ y^{(1)}(0) &= \alpha_1 \end{aligned}$$

$$y = c_1 \cos(\beta x) + c_2 \sin(\beta x)$$

$$c_1 = \alpha_0$$

$$y^{(1)}(0) = -\beta c_1 \cancelto{0}{\sin(\beta 0)} + \beta c_2 \cancelto{1}{\cos(\beta 0)}$$

$$= \beta c_2 = \alpha_1$$

$$c_2 = \frac{\alpha_1}{\beta}$$

$$y = \alpha_0 \cos(\beta x) + \frac{\alpha_1}{\beta} \sin(\beta x)$$

example Damped Oscillator

$$\ddot{y} + 2\beta\dot{y} + \alpha^2 y = 0$$

$$\dot{y} = \frac{dy}{dt}$$

BCs

$$\left.\frac{dy}{dt}\right|_{t=0} = 0$$

$$\left.y\right|_{t=0} = 1$$

Assume a solution $y \sim e^{-i\omega t}$

$$-\omega^2 - i\omega 2\beta + \alpha^2 = 0$$

$$\omega^2 + 2i\beta\omega - \alpha^2 = 0$$

$$\omega = \frac{-\cancel{2}i\beta \pm \sqrt{\cancel{4}\beta^2 + \cancel{4}\alpha^2}}{\cancel{2}}$$

$$\omega = -i\beta \pm \omega_0 \quad \text{with } \omega_0 = \sqrt{\beta^2 + \alpha^2}$$

$$y \sim e^{-\beta t} e^{\pm i\omega_0 t}$$

$$y = [A\cos\omega_0 t + B\sin\omega_0 t]e^{-\beta t}$$

$$y(0) = A = 1$$

$$\dot{y}(0) = -\beta A + \omega_0 B = 0$$

$$B = \frac{\beta}{\omega_0} \Rightarrow y = \left[\cos\omega_0 t + \frac{\beta}{\omega_0}\sin\omega_0 t\right]e^{-\beta t}$$

Example Boundary Value Problem (Dirichlet)

$$\frac{d^2y}{dx^2} + \beta^2 y = 0 \quad \Rightarrow \text{specify } y \text{ at } x = 0, 1$$

$$y = c_1 \cos \beta x + c_2 \sin \beta x$$

$$y(0) = \alpha_0 \quad \Rightarrow \quad c_1 = \alpha_0$$

$$y(1) = \alpha_1 \quad \Rightarrow \quad y(1) = \alpha_0 \cos \beta + c_2 \sin \beta = \alpha_1$$

$$c_2 = \frac{\alpha_1 - \alpha_0 \cos \beta}{\sin \beta}$$

Solution ok except $\beta = \pi$

$$y = c_1 \cos \pi x + c_2 \sin \pi x$$

$\Rightarrow$ $\sin(\pi x)$ does not contribute at 0, 1 so only have solution for $\alpha_0 = \alpha_1$

$\Rightarrow$ not all boundary value problems have solutions