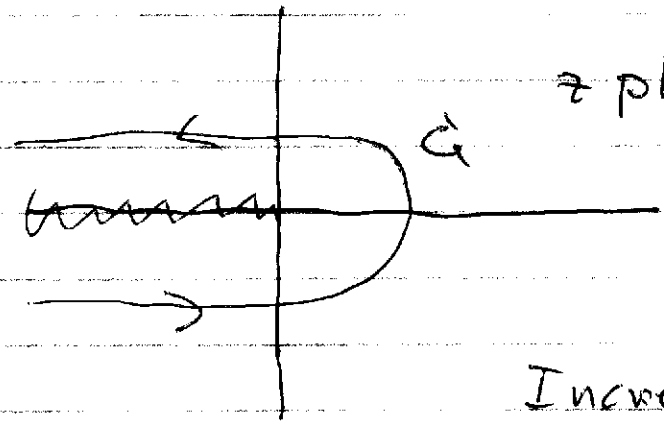
Analytic continuation of $I_\nu(s)$ to negative s

$$I_\nu(s) = \frac{1}{2\pi i} \int_C e^{\frac{s}{2}\left(z+\frac{1}{z}\right)} \frac{dz}{z^{\nu+1}} \qquad \mathrm{Re}(s) > 0$$



z plane

$-\pi < \mathrm{Arg}\, z < \pi$

Let $s = \alpha e^{i\beta}$ with α real.

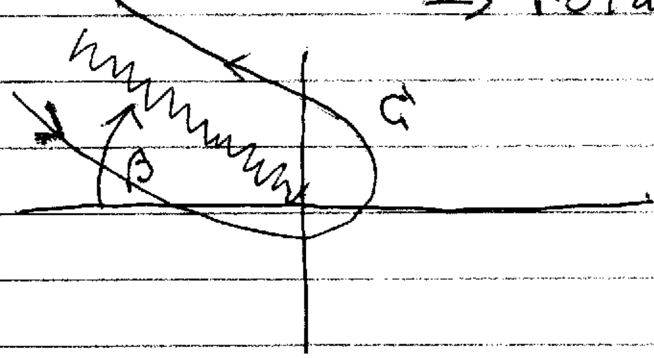
Increase β from 0 to π, making sure that the integral converges for each value of β.

$$I_\nu(\alpha e^{i\beta}) = \frac{1}{2\pi i} \int_C \frac{dz}{z^{\nu+1}} e^{\frac{\alpha e^{i\beta}}{2}\left(z+\frac{1}{z}\right)}$$

Can increase β and integral remains bounded until $\beta = \frac{\pi}{2}$

For large negative z ~~want~~ $e^{\frac{\alpha}{2} e^{i\beta} z} \to 0$ even when $\beta > \pi/2$. How?

$\Rightarrow$ rotate the cut and C



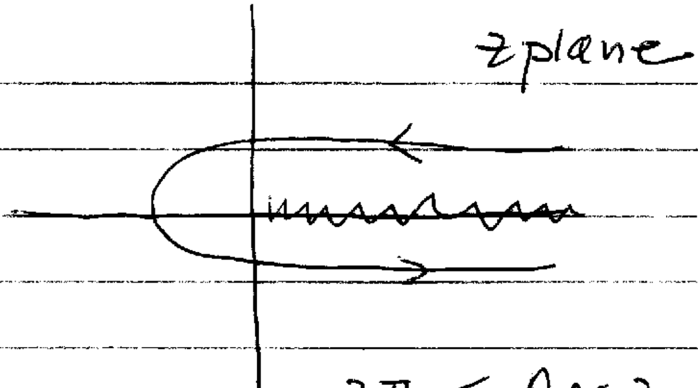
On top of cut: $\mathrm{Arg}(z) = i\pi - \beta$

Below cut: $\mathrm{Arg}(z) = -i\pi - \beta$

$-\pi - \beta < \mathrm{Arg}(z) < \pi - \beta$

$\beta \to \pi$

$\Longrightarrow$

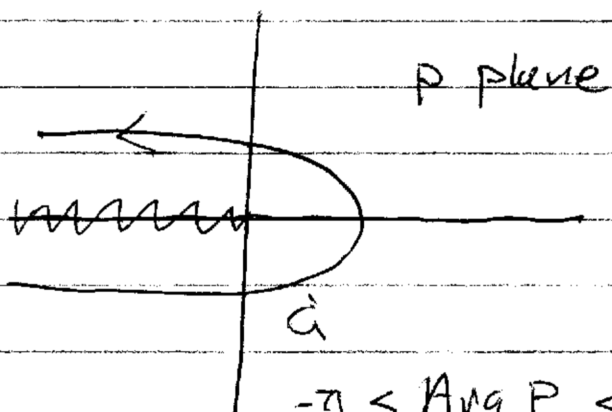


$-2\pi < \operatorname{Arg} z < 0$

Let $p = z e^{i\pi}$ $\Rightarrow$ chosen so

$-\pi < \operatorname{Arg}(p) < \pi$

$$I_\nu(\alpha e^{i\pi}) = \frac{1}{2\pi i} \int_C dp\, e^{-i\pi} \frac{e^{\frac{\alpha}{2} e^{i\pi}\left(p e^{-i\pi} + \frac{1}{p e^{-i\pi}}\right)}}{\left(p e^{-i\pi}\right)^{\nu+1}}$$



$-\pi < \operatorname{Arg} P < \pi$

$$= \frac{e^{i\pi\nu}}{2\pi i} \int dp\, \frac{e^{\frac{\alpha}{2}\left(p + \frac{1}{p}\right)}}{p^{\nu+1}}$$

$$= e^{i\pi\nu} I_\nu(\alpha)$$

$$\boxed{I_\nu(\alpha e^{i\pi}) = e^{i\pi\nu} I_\nu(\alpha)}$$