Approximating Contour Integrals

Saddle Point Methods $\Rightarrow$ require a large parameter.

$$\Gamma(p) = \int_0^\infty dz\, e^{-z} z^{p-1} \quad \Rightarrow \text{gamma function}$$

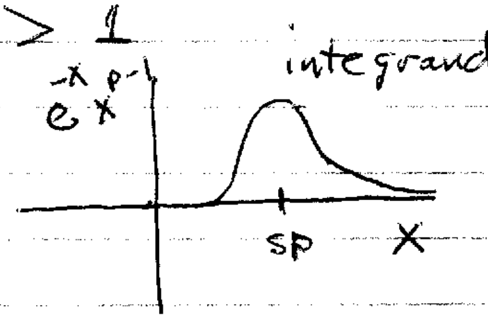
$\Rightarrow$ defined for $Re(p) > 0$

$\Gamma(p) = (p-1)\Gamma(p-1)$

$\Gamma(n) = (n-1)!$ for n integer. why?

Take p to be real for simplicity and take $p >> 1$

$$\Gamma(p) = \int_0^\infty dz\, e^{-z+(p-1)\ln z}$$



Let $h(z) = (p-1)\ln(z) - z$

Integrand peaks where

$$h'(z) = 0 = (p-1)\frac{1}{z} - 1$$

$$z_{sp} = p-1$$

$\Rightarrow$ How do we know that z_{sp} must be a saddle point?

$$h'' = -\frac{p-1}{z^2} \Rightarrow h''(z_{sp}) = -\frac{(p-1)}{(p-1)^2} = -\frac{1}{p-1}$$

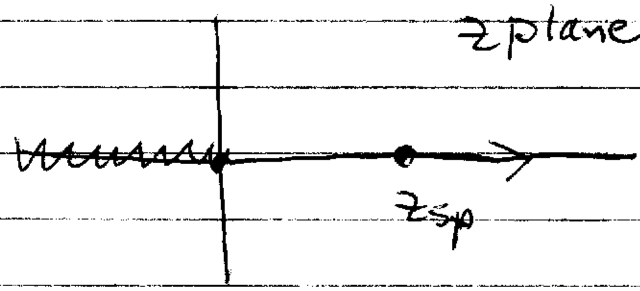
Near the saddle point expand h in a Taylor series

$$h(z) \simeq h(z_{sp}) + \frac{1}{2} h''(z_{sp}) \left(z - z_{sp}\right)^2$$

Let $\Delta z = z - z_{sp}$

$$h(z) \simeq (p-1)\ln(p-1) - (p-1) - \frac{1}{2} \frac{\Delta z^2}{p-1}$$

Near z_{sp}: $$e^{h(z)} \simeq (p-1)^{p-1} e^{-(p-1)} e^{-\frac{1}{2}\frac{\Delta z^2}{p-1}}$$



note: $h(z)$ goes to zero away from S.P. along the real axis

B.C. need because p might not be an integer

$h(z)$ increases away from S.P. along y axis

$\Rightarrow \Delta z = i\Delta y$

$\Rightarrow$ as expected for S.P.

Approximate $\Gamma(p)$ as

$$\Gamma(p) \simeq (p-1)^{p-1} e^{-(p-1)} \int_{-\infty}^{\infty} d\Delta z\, e^{-\frac{1}{2}\frac{\Delta z^2}{p-1}}$$

$\Rightarrow$ where extend integral over Δz to $\pm\infty$ since when Taylor series breaks down, the integrand is already small

$$\Gamma(p) = (p-1)^{p-1} e^{-(p-1)} \sqrt{\pi} \left[2(p-1)\right]^{\frac{1}{2}}$$

$$\boxed{\Gamma(p) = \sqrt{2\pi}\,(p-1)^{p-\frac{1}{2}} e^{-(p-1)}}$$ Stirlings formula

$\Rightarrow$ very accurate for large p.

Check: Is the next order term in the Taylor series small when

$$\frac{1}{2} h'' \Delta z^2 \sim 1 \quad \Rightarrow \text{where}$$

$$\Delta z \sim (p-1)^{1/2} \qquad e^{h(z)} \rightarrow 0$$

next term

$$\frac{1}{3!} h''' \Delta z^3 = \frac{1}{6} \frac{2(p-1)}{z_{sp}^3} \Delta z^3$$

$$\sim \frac{\Delta z^3}{(p-1)^2} \sim \frac{1}{(p-1)^{1/2}} \ll 1$$

$\Rightarrow$ valid for

$$p-1 \gg 1$$

Now allow p to be complex.

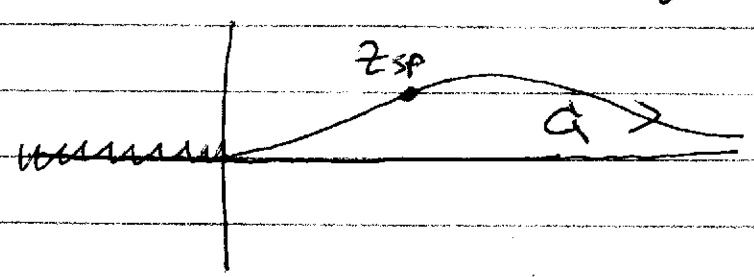
$\Rightarrow$ still require $\mathrm{Re}(p) > 0$

S.P as before: $z_{sp} = (p-1) \equiv q$

$$= |q| e^{i\beta}$$

S.P. in the complex plane

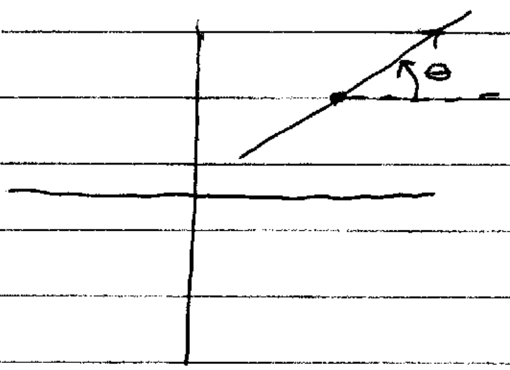
~~$\Rightarrow$ move the contour to a path through the S.P~~



Again expand $h(z)$ around z_{sp}

$$\Gamma(P) = \int_{C} dz\, e^{-\frac{(z-z_{sp})^2}{2(P-1)}} (P-1)^{P-1} e^{-(P-1)}$$

What is the angle of C through the s.p.?



$$\Delta z = s\, e^{i\theta}$$

with s real

Choose θ so that

$$\frac{(z-z_{sp})^2}{2(P-1)} \text{ is real and positive}$$

$$\Rightarrow \frac{e^{2i\theta}}{P-1} \sim \frac{e^{2i\theta}}{e^{i\beta}} = e^{i(2\theta-\beta)}$$

$$\Rightarrow \theta = \frac{\beta}{2} = \frac{1}{2}\operatorname{Arg}(P-1)$$

This defines the <u>path of steepest descent</u>

$\Rightarrow$ PSD

$\Rightarrow$ path along which $e^{h(z)}$ goes to zero the fastest

$$e^{h(z)} \simeq (P-1)^{P-1} e^{-(P-1)} e^{-\frac{s^2}{|\mathcal{E}|}\frac{1}{2}}$$

$$|\mathcal{E}| = |P-1|$$

$$\Gamma(p) = (p-1)^{p-1} e^{-(p-1)} \int_{-\infty}^{\infty} e^{-\frac{s^2}{2|p-1|}} \, ds \, e^{i\theta}$$

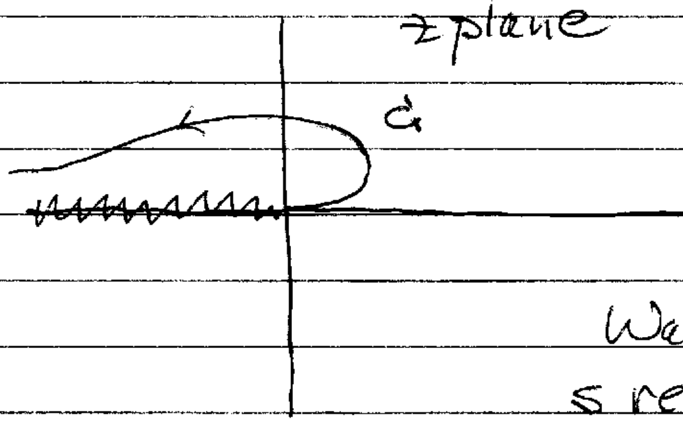
$$= (p-1)^{p-1} e^{-(p-1)} \sqrt{2\pi} \, \underbrace{|p-1|^{1/2} e^{i\frac{1}{2}\mathrm{Arg}(p-1)}}_{(p-1)^{1/2}}$$

$$= \sqrt{2\pi} \, (p-1)^{p-\frac{1}{2}} e^{-(p-1)}$$

$\Rightarrow$ same as before

Asymtotic form of the Hankel function

$$H_\nu^{(1)}(s) = \frac{1}{\pi i}\int_{0\,C_1}^{-\infty} dz\, \frac{e^{\frac{s}{2}\left(z-\frac{1}{z}\right)}}{z^{\nu+1}}$$



Have a cut beginning at the BP at $z=0$ when $\nu \neq$ integer.

Want to evaluate $H_\nu^{(1)}$ for s real and large and $\nu \sim 1$.

Basic idea: The integrand remains small over most of the path of integration if it is deformed to pass through a saddle point and follows the PSD through the S.P.

$\Rightarrow$ contribution over the S.P. dominates the integral.

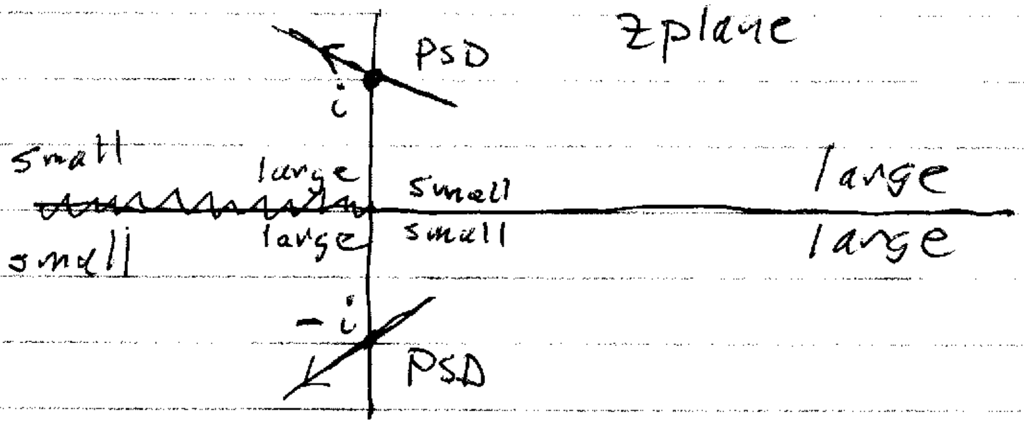
Essential point: Explore the structure of the integrand in the complex plane to determine whether the S.P.s dominate the integral and if so which S.P. is most important.

Locate the saddle points:

$$h(z) = \frac{s}{2}\left(z - \frac{1}{z}\right) \quad \Rightarrow \text{ignore } \frac{1}{z^{\nu+1}}. \text{ Why?}$$

$$h' = \frac{5}{2}\left(1 + \frac{1}{z^2}\right) = 0 \quad \Rightarrow \quad z_{sp} = \pm i$$

Topology of $e^{\frac{5}{2}\left(z - \frac{1}{z}\right)}$ $\qquad h'' = -5\,\frac{1}{z^3}$



Move the contour C to pass through $z_{sp} = i$

Why is $z_{sp} = -i$ unimportant?

* Expand $h(z)$ around $z_{sp} = i$

$$h(z) \simeq h(z_{sp}) + \frac{1}{2} h''(z_{sp})\left(z - z_{sp}\right)^2$$

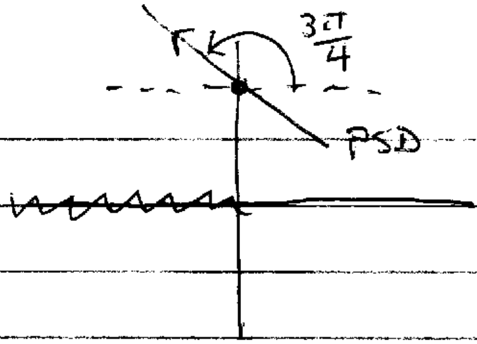
$$= \frac{5}{2}\left(i - \frac{1}{i}\right) + \frac{1}{2}\left(-5\,\frac{1}{i^3}\right)(z-i)^2$$

$$= i5 - i\,\frac{5}{2}(z-i)^2 \qquad \cancel{z-i} = q e^{i\theta}$$

$$= i5 - i\,\frac{5}{2} q^2 e^{2i\theta}$$

$$\text{PSD} \Rightarrow i e^{2i\theta} = 1$$

$$e^{2i\theta} e^{i\frac{\pi}{2}} = 1 \quad \Rightarrow \quad \theta = -\frac{\pi}{4}, \frac{3\pi}{4}$$



$$\Rightarrow \theta = \frac{3\pi}{4}$$

$$dz = dq\, e^{i\theta}$$

$$H_\nu^{(1)}(s) \simeq \frac{1}{\pi i} \int_{-\infty}^{\infty} dq\, \frac{e^{i\frac{3\pi}{4}}\, e^{is}}{i^{\nu+1}}\, e^{-\frac{s}{2}q^2}$$

$$i^{\nu+1} = e^{i\frac{\pi}{2}(\nu+1)} \quad \Rightarrow z_{sp} = e^{i\frac{\pi}{2}}$$

$\Rightarrow$ because of B.C.

$$H_\nu^{(1)}(s) \simeq \frac{1}{\pi i}\, e^{-i\frac{\pi}{2}(\nu+1)}\, e^{i\frac{3\pi}{4}}\, e^{is} \underbrace{\int_{-\infty}^{\infty} dq\, e^{-\frac{s}{2}q^2}}_{\sqrt{\pi}\sqrt{\frac{2}{s}}}$$

$$\boxed{H_\nu^{(1)}(s) \simeq -\sqrt{\frac{2}{\pi s}}\, e^{\frac{3\pi i}{4}}\, e^{is}\, e^{-i\frac{\pi}{2}\nu}}$$

For $s \gg 1$