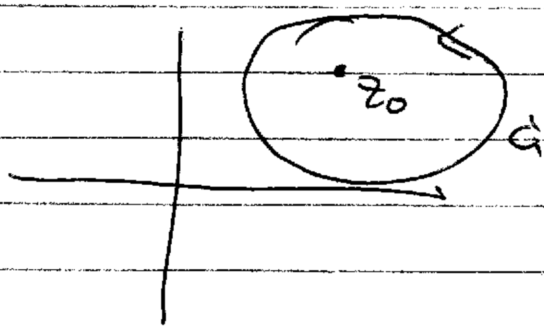
Residues

Consider a function $f(z)$ with an isolated singular point at z_0 so that

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$



$$\oint_C dz \, (z - z_0)^n = \underbrace{\begin{matrix} 0 & n \neq -1 \\ 2\pi i & n = -1 \end{matrix}}_{\text{homework}}$$

$$\Rightarrow \oint_C f(z) \, dz = 2\pi i \, a_{-1}$$

$$a_{-1} = \frac{1}{2\pi i} \oint_C dz \, f(z)$$

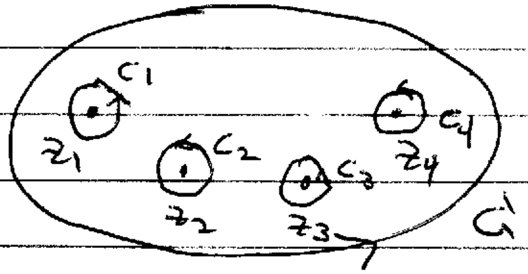
a_{-1} is the residue of $f(z)$ at $z = z_0$.

Residue Theorem

Consider a function $f(z)$ which has a finite # of isolated singularities z_j within a closed contour C. Then

$$\oint_C dz \, f(z) = 2\pi i \sum_j a_{-1}(z_j)$$

$$= 2\pi i \text{ sum of residues}$$



Shrink contour C around the singularities $\Rightarrow$ why is this ok?

$$\oint_C dz\, f(z) = \sum_j \oint_{C_j} dz\, f(z)$$

$$= \sum_j 2\pi i \sum_j a_{-1}(z_j)$$

<u>How can you calculate</u> $a_{-1}(z_j)$?

Suppose $f(z)$ has an mth order pole at z_j.
Then $g(z) = (z - z_j)^m f(z)$ is analytic at z_j.

$$\oint_{C_j} dz\, f = \oint dz \frac{g(z)}{(z - z_j)^m} = \frac{2\pi i}{(m-1)!}\, g^{(m-1)}(z_j)$$

$$a_{-1}(z_j) = \frac{1}{(m-1)!} \left[(z - z_j)^m f(z) \right]^{(m-1)} \Bigg|_{z = z_j}$$

If $f(z)$ has an essential singularity at z_j, expand $f(z)$ in Laurent series
$\Rightarrow$ yields $a_{-1}(z_j)$

Alternate method for 1st order pole

Suppose $f(z) = \dfrac{P(z)}{Q(z)}$

where $P(z)$, $Q(z)$ analytic at z_j but
$Q(z_j) = 0$

$\Rightarrow$ expand $Q(z)$ in a Taylor series around z_j

$$Q(z) \simeq Q'(z_j)(z-z_j) + \cdots$$

with $Q'(z_j) \neq 0$

$$f(z) \simeq \frac{P(z_j)}{Q'(z_j)} \frac{1}{z-z_j}$$

$$a_{-1} = \frac{P(z_j)}{Q'(z_j)}$$

example: $\quad \cancel{f(z)} = \dfrac{\cos z}{e^z - 1}$

First order pole at $z=0$,

$$Q(z) \simeq \cancel{Q(0)}^{\,0} + Q'z = z$$

$$P(0) = 1$$

$$a_{-1} = \frac{1}{1} = 1$$

example: $f(z) = \dfrac{\cos z}{e^{z^2} - 1}$

$\Rightarrow$ second order pole at $z = 0$

$$a_{-1} = \frac{1}{1!}\left[\frac{z^2 \cos z}{e^{z^2}-1}\right]'\Bigg|_{z=0}$$

$$= \left(\frac{z^2 \cos z}{\cancel{1} + z^2 + \frac{1}{2}z^4 - \cancel{1}}\right)'\Bigg|_{z=0} = \left(\frac{\cos z}{1+\frac{1}{2}z^2}\right)'\Bigg|_{z=0}$$

$$= 0$$

Evaluation of integrals

(1) $I = \int_{-\infty}^{\infty} dx\, f(x)$ with $f = \frac{P(x)}{Q(x)}$

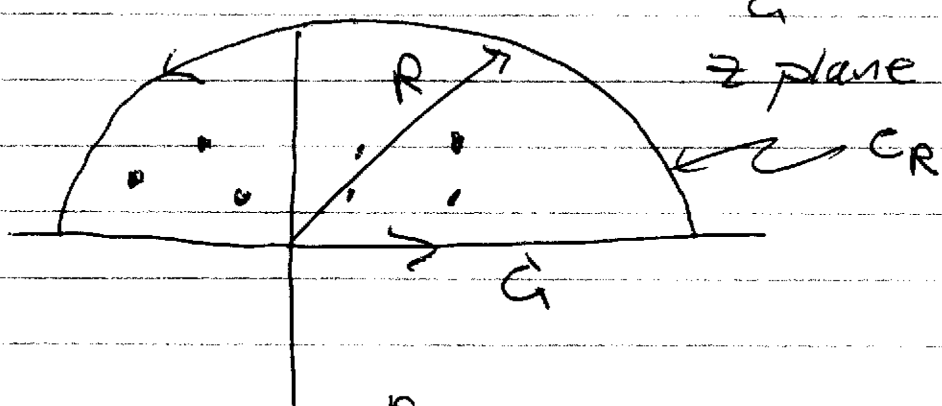
with P, Q polynomials with degree Q − degree P ≥ 2

$\Rightarrow$ $f(x)$ has no poles on the real axis.

$\Rightarrow$ treat x as a complex variable z.

$I = \int dz\, f(z)$

Consider $I_c = \int_{C_1} dz\, f(z)$



$$I_c = \int_{-R}^{R} dz\, f(z) + \int_{C_R} dz\, f(z)$$

$$\int_{C_R} dz\, f(z) = iR \int_0^{\pi} d\theta\, e^{i\theta} f(Re^{i\theta})$$

$z = Re^{i\theta}$, $dz = i\, d\theta\, Re^{i\theta}$

$R f(Re^{i\theta}) \to 0$ as $R \to \infty$

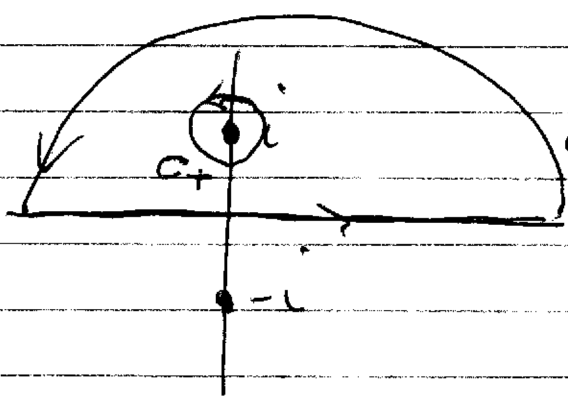
$$\Longrightarrow \int_{C_R} dz\, f = 0$$

$$\lim_{R\to\infty} \int_{-R}^{R} dz\, f(z) = I = \oint_{C} dz\, f(z)$$

$$I = 2\pi i \left[\sum \text{residues in the upper half plane} \right]$$

What if closed contour in lower half plane?

example: $I = \int_{-\infty}^{\infty} dz \frac{1}{1+z^2}$



$$= \oint_{C} dz \frac{1}{1+z^2}$$

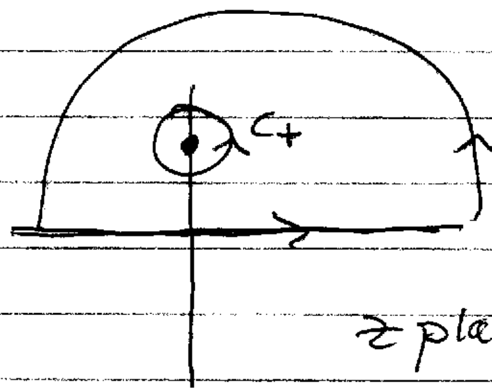
$$= \oint_{C_+} dz \frac{1}{(z+i)(z-i)}$$

$$= \frac{1}{2i} \oint_{C_+} \frac{dz}{(z-i)} = \frac{2\pi i}{2i}$$

$$= \pi$$

$$\int_{-\infty}^{\infty} dz \frac{1}{1+z^2} = \pi$$

example: $I = \int_{-\infty}^{\infty} dz \, \frac{1}{(1+z^2)^2}$



$$= \int_C dz \, \frac{1}{(z-i)^2 (z+i)^2}$$

$$= \oint_{C_+} dz \, \frac{1}{(z-i)^2 (z+i)^2}$$

$\Rightarrow$ second order pole at $z = i$

$\Rightarrow$ use Cauchy's derivative formula

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint dz \, \frac{f(z)}{(z-z_0)^{n+1}} \qquad \frac{1}{(z+i)^2}$$

$z_0 = i \qquad n = 1$

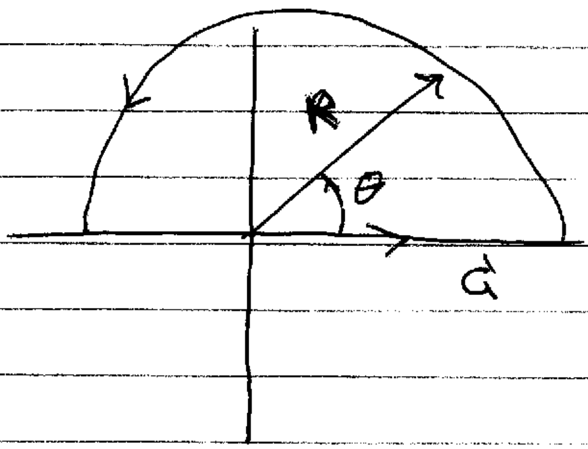
$$I = \frac{2\pi i}{1!} \frac{d}{dz}\left[\frac{1}{(z+i)^2}\right]\Bigg|_{z=i} = 2\pi i \, \frac{-2}{(z+i)^3}\Bigg|_{z=i}$$

$$= 2\pi i \, \frac{-2}{8 i^3} = \frac{\pi}{2}$$

(2) Consider integrals of the form

$$I = \int_{-\infty}^{\infty} dx\, e^{ikx} f(x) \qquad \text{with } k \text{ real and } k > 0$$

$$= \int_{-\infty}^{\infty} dz\, e^{ikz} f(z)$$



z plane

$\Rightarrow$ note that

$$e^{ikz} = e^{ikx} e^{-ky}$$

$\longrightarrow 0$ as $y \to \infty$ in the UHP

Let

$$I_C = \oint_C dz\, e^{ikz} f(z)$$

$$= \int_{-R}^{R} dz\, e^{ikz} f(z) + \underbrace{Ri \int_0^{\pi} d\theta\, e^{ikRe^{i\theta}} e^{i\theta} \otimes f(Re^{i\theta})}_{I_R}$$

$$|I_R| \leq R \int_0^{\pi} d\theta\, e^{-kR\sin\theta} |f(Re^{i\theta})|$$

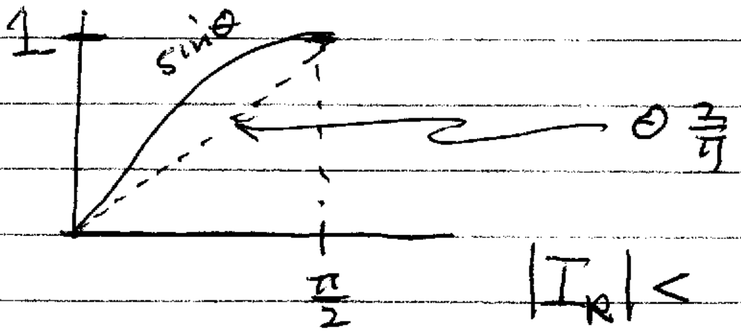
Assume $f(Re^{i\theta}) \to 0$ as $R \to \infty$

$$\Rightarrow |f(\,)| < \epsilon$$

$$|I_R| < \epsilon R \int_0^{\pi} d\theta\, e^{-kR\sin\theta}$$

$\sin\theta$ symmetric around $\pi/2$

$$|I_R| \le 2\epsilon R \int_0^{\pi/2} d\theta\, e^{-kR\sin\theta} < 2\epsilon R \int_0^{\pi/2} d\theta\, e^{-\frac{kR2\theta}{\pi}}$$



$$\theta \frac{2}{\pi} < \sin\theta$$

$$|I_R| < 2\epsilon R \frac{\pi}{2kR}\left(1 - \cancelto{0}{e^{-kR}}\right)$$

$$= \frac{\epsilon\pi}{k}$$

$$\lim_{R\to\infty} |I_R| = 0$$

$$\Rightarrow I = \oint_C dz\, e^{ikz} f(z)$$

$\Rightarrow$ can close contour in UHP
as long as $f \to 0$ as $z \to \infty$

$\Rightarrow$ Jordan's Lemma

$\Rightarrow$ if $k<0$ close in LHP

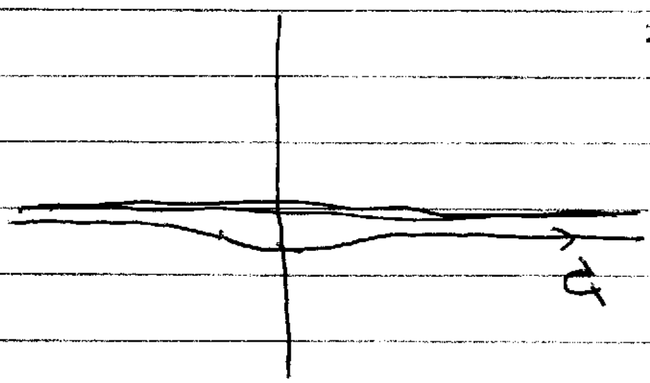
example: $I = \int_0^\infty dx \, \frac{\sin x}{x}$

$\Rightarrow$ note that no singularity at $x = 0$

$I = \frac{1}{2} \int_{-\infty}^{\infty} dz \, \frac{\sin z}{z}$ since even function on real axis

First deform contour away from $z = 0$

$\Rightarrow$ why is this correct?



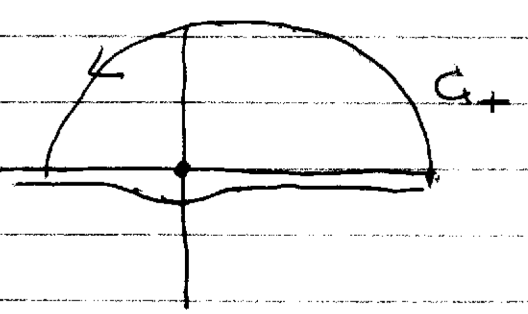
$I = \frac{1}{2} \int_C dz \, \frac{\sin z}{z}$

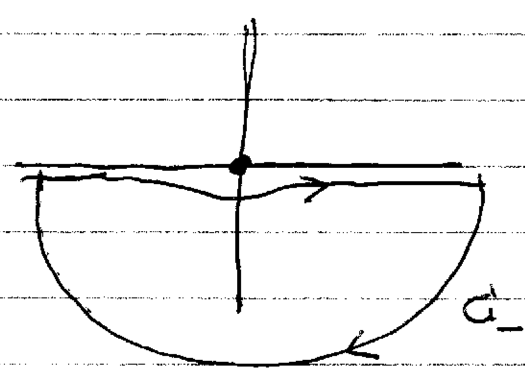
$\Rightarrow$ now can write

$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$

$I = \underbrace{\frac{1}{4i} \int_C dz \, \frac{e^{iz}}{z}}_{\text{close in UHP}} - \underbrace{\frac{1}{4i} \int dz \, \frac{e^{-iz}}{z}}_{\text{close in LHP}}$

Jordan's Lemma $\rightarrow$ close in UHP





$$I = \frac{1}{4i}\oint_{C_+} dz\, \frac{e^{iz}}{z} - \underbrace{\frac{1}{4i}\oint_{C_-} dz\, \frac{e^{-iz}}{z}}_{0 \;\Rightarrow\; \text{no enclosed singularities}}$$

$$= \frac{1}{4i}\, 2\pi i$$

$$\boxed{I = \frac{\pi}{2}}$$

(3) <u>Trig functions</u> (finite integrals)

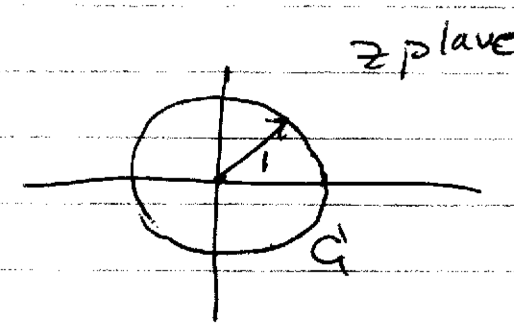
$$I = \int_0^{2\pi} d\theta\, f(\sin\theta, \cos\theta)$$

Let $z = e^{i\theta}$, $dz = i\, d\theta\, e^{i\theta}$

$$= z\, i\, d\theta$$

$$\sin\theta = \frac{z - \frac{1}{z}}{2i} \qquad d\theta = \frac{dz}{iz}$$

$$\cos\theta = \frac{z + \frac{1}{z}}{2}$$



$$I = -i \oint_{C_1} \frac{dz}{z} f(z)$$

$\Rightarrow$ same as polynomial integrals

4 Exponentials

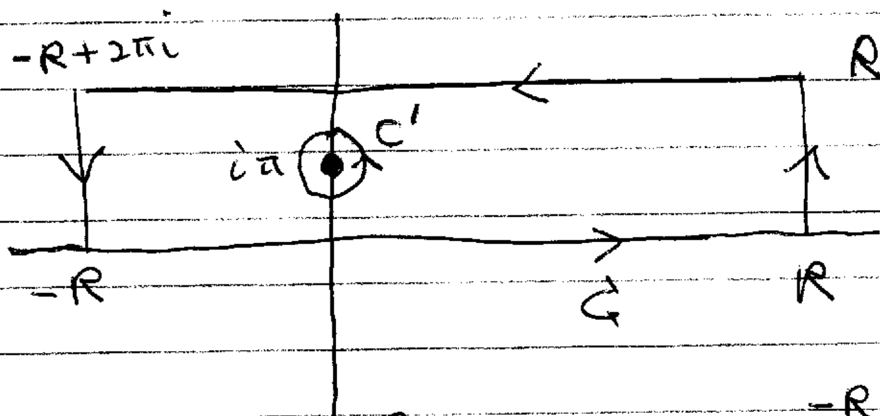
$$I = \int_{-\infty}^{\infty} dx \, \frac{e^{ax}}{1+e^{x}} \qquad 0<a<1$$

$\Rightarrow$ integrand goes to zero at $x \pm \infty$.

Consider

$$I_c = \int_C dz \, \frac{e^{az}}{1+e^{z}}$$

$\Rightarrow$ note that the denominator is unchanged by $2\pi i$ shift.



$\Rightarrow$ contributions from ends goes to zero as $R \to \infty$.

$$I_c = \int_{-R}^{R} dz \, \frac{e^{az}}{1+e^{z}} + \int_{R}^{-R} dz \, \frac{e^{a(z+2\pi i)}}{1+e^{z}}$$

$$= \int_{-R}^{R} dz \, \frac{e^{az}}{1+e^{z}} \left(1 - e^{2\pi i a}\right) = I\left(1-e^{2\pi i a}\right)$$

Singularity at $z_0 = i\pi$

$\Rightarrow$ shrink contour to C'

$\Rightarrow$ first order pole at $z = i\pi$

$$z = i\pi + \Delta z$$

$$I_\epsilon = \int_{c'} dz \frac{e^{az}}{1+e^{z}} = \oint_{c'} d\Delta z \frac{e^{ai\pi}}{1-e^{\Delta z}}$$

$$= -e^{i\pi a} \, 2\pi i$$

$$I = \frac{-2\pi i e^{i\pi a}}{1-e^{2\pi i a}} = \frac{2\pi i}{e^{\pi i a} - e^{-i\pi a}} = \frac{\pi}{\sin \pi a}$$