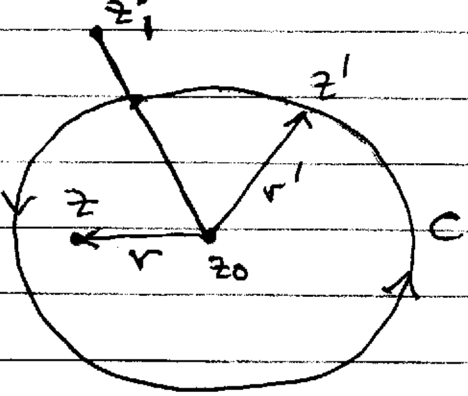
Series and Poles

Taylor Series

Want to expand $f(z)$ around z_0 where z_1 is the nearest point where $f(z)$ is not analytic.



C is a circle of radius r' from z_0.

$$f(z) = \frac{1}{2\pi i}\oint_C dz' \frac{f(z')}{z'-z} = \frac{1}{2\pi i}\oint_C dz' \frac{f(z')}{z'-z_0+z_0-z}$$

$$= \frac{1}{2\pi i}\oint_C dz' \frac{f(z')}{(z'-z_0)\left(1+\frac{z_0-z}{z'-z_0}\right)}$$

Let $t = \frac{z-z_0}{z'-z_0}$, $|t| = \frac{r}{r'} < 1$

$$\frac{1}{1-t} = 1 + t + t^2 + \cdots t^{N-1} + \frac{t^N}{1-t}$$

$\Rightarrow$ note that this is exact

$$f(z) = \frac{1}{2\pi i}\oint dz' \frac{f(z')}{z'-z_0}\left[\sum_{n=0}^{N-1}\left(\frac{z-z_0}{z'-z_0}\right)^n + \frac{(\ )^N}{1-(\ )}\right]$$

$$= \frac{1}{2\pi i}\sum_{n=0}^{N-1}(z-z_0)^n \oint_C dz' \frac{f(z')}{(z'-z_0)^{n+1}} + R_N$$

$$f(z) = \sum_{n=0}^{N-1} \frac{(z-z_0)^n}{n!} f^{(n)}(z_0) + R_N$$

$$R_N = \frac{1}{2\pi i} \oint_C dz' \frac{f(z')}{(z'-z_0)^{N+1}} \frac{(z-z_0)^N}{z'-z}(z'-z_0)$$

$$|R_N| \lesssim \frac{1}{2\pi} \oint_C |dz'||f(z')| \frac{r^N}{r'^N(r'-r)} \lesssim |f(z)|^C_{max} \left(\frac{r}{r'}\right)^N \frac{r'}{r'-r}$$

$|f(z)|^C_{max}$ = maximum value of f on C

Since $\frac{r}{r'} < 1$, $\lim_{N\to\infty} R_N = 0$

$$\boxed{f(z) = \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{n!} f^{(n)}(z_0)}$$ Taylor series

$\Rightarrow$ converges only where

$$|z-z_0| < |z_1 - z_0|$$

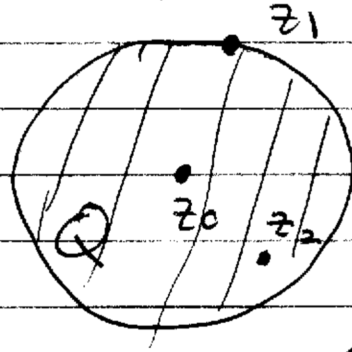
since $|f(z)|^C_{max} \longrightarrow \infty$ if C includes z_1

example: Expand $\frac{1}{z-3}$ around $z_0 = 2.5$

$\Rightarrow$ radius of convergence $|z-z_0| < 0.5$

Analytic Continuation

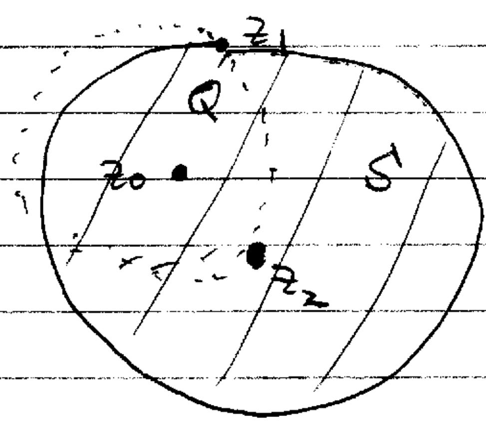
Consider an analytic function $f(z)$ with a singularity at z_1. Expand $f(z)$ in a Taylor series around z_0.



The expansion is valid in domain Q.

Consider a point z_2 within Q. Calculate $f(z_2)$ and all of its derivatives at z_2.

$\Rightarrow$ Can expand f in a Taylor series around z_2.



Region of convergence of Taylor series around z_2 is now S.

$\Rightarrow$ can now calculate f over a larger domain

$\Rightarrow$ continue until can evaluate f everywhere except where non analytic

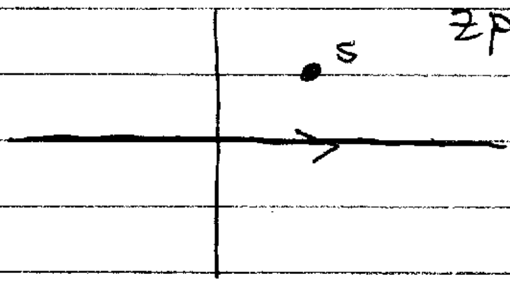
$\Rightarrow$ the process of extending a region where f can be evaluated is called analytic continuation.

$\Rightarrow$ ~~note that the series around z_2 is~~ not ~~the same as around z_0.~~

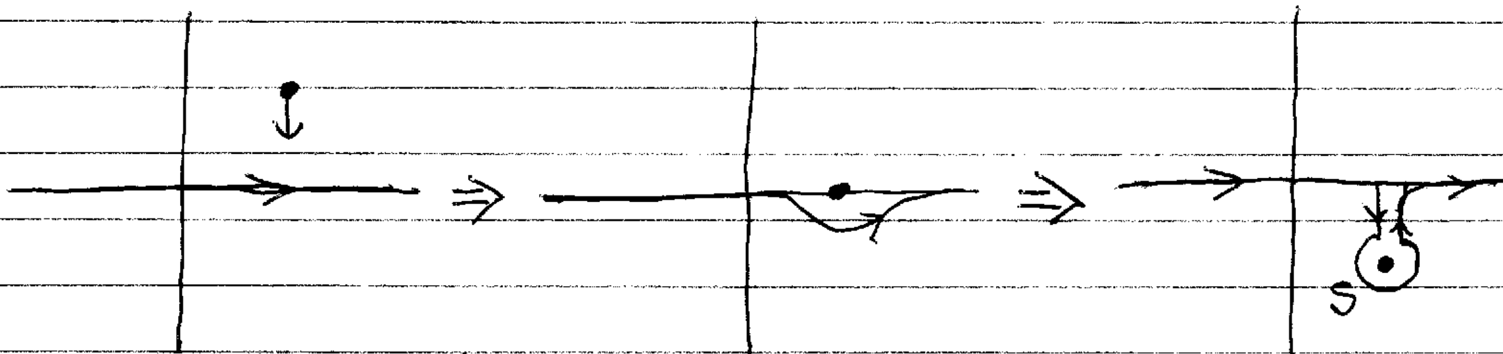
example: plasma dispersion function

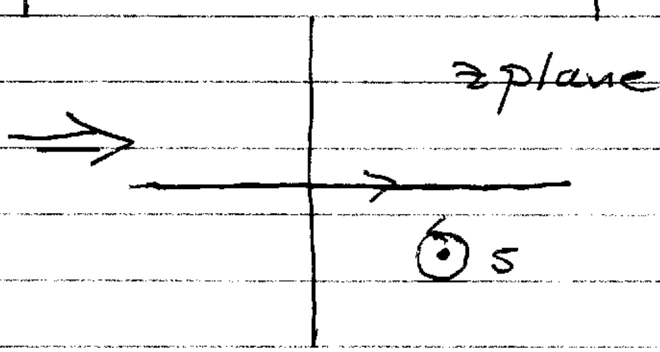
$$Z(s) = \int_{-\infty}^{\infty} dz \, \frac{e^{-z^2}}{z-s} \qquad \text{where } \operatorname{Im}(s) > 0$$

$\Rightarrow$ from causality



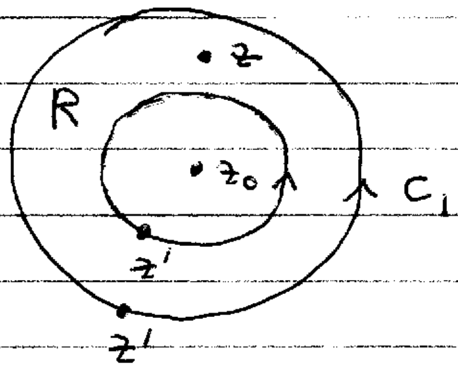
Can analytically extend to the lower half plane





Laurent Series

Let $f(z)$ be analytic on and between two circles C_1 and C_2 but can be non-analytic inside C_2. Want a series representation for f around z_0 where f could be singular



$$f(z) = \frac{1}{2\pi i}\oint_{C_1} dz' \frac{f(z')}{z'-z} - \frac{1}{2\pi i}\oint_{C_2} dz' \frac{f(z')}{z'-z}$$

$\Rightarrow$ since analytic in R

$$C_1 \left\{ \frac{1}{z'-z} = \frac{1}{z'-z_0-(z-z_0)} = \frac{1}{z'-z_0}\,\frac{1}{1-\left(\frac{z-z_0}{z'-z_0}\right)} \right.$$

$$= \frac{1}{z'-z_0}\sum_{n=0}^{\infty}\left(\frac{(z-z_0)}{z'-z_0}\right)^n$$

$$C_2 \left\{ \frac{1}{z'-z} = \frac{-1}{z-z_0-(z'-z_0)} = -\frac{1}{z-z_0}\,\frac{1}{1-\frac{z'-z_0}{z-z_0}} \right.$$

$$= -\frac{1}{z-z_0}\sum_{n=0}^{\infty}\left(\frac{z'-z_0}{z-z_0}\right)^n$$

$$f(z) = \sum_{n=0}^{\infty} (z-z_0)^n \overbrace{\frac{1}{2\pi i} \oint_{C_1} dz' \frac{f(z')}{(z'-z_0)^{n+1}}}^{a_n}$$

$$+ \sum_{n=0}^{\infty} \frac{1}{(z-z_0)^{n+1}} \underbrace{\frac{1}{2\pi i} \oint_{C_2} dz' f(z') (z'-z_0)^n}_{b_{n+1}}$$

$$= \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n \frac{1}{(z-z_0)^n}$$

$\Rightarrow$ since $f(z)$ not analytic within C_1, C_2, can't use Cauchy's integral formula to evaluate a_n, b_n

$\Rightarrow$ when $f(z)$ analytic within C_2, $b_n = 0$ for all n and series reduces Taylor series.

$\Rightarrow$ integrals for a_n, b_n can be evaluated, along any contour in R $\Rightarrow$ since analytic

$\Rightarrow$ rewrite series

$$f = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

$$a_n = \frac{1}{2\pi i} \oint_C dz' \frac{f(z')}{(z'-z_0)^{n+1}}$$

with C between C_1, C_2.

$\Rightarrow$ easiest approach is not always to do the integral.

Convergence of series
(Arfken 7th Edition Ch 1)

Brief review of the convergence of series of complex numbers

$$\text{Let } S = \sum_{n=0}^{\infty} z_n = \sum_n (x_n + i y_n)$$
$$= X + i Y$$

with x_n, y_n real.

$$S_N = \sum_{n=1}^{N} z_n = X_N + i Y_N$$

1) A necessary and sufficient condition that
$$\lim_{N\to\infty} S_N = S$$
is that
$$\lim_{N\to\infty} \begin{pmatrix} X_N \\ Y_N \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix}$$

2) A necessary condition for a series of real numbers to converge is that
$$\lim_{n\to\infty} \begin{pmatrix} x_n \\ y_n \end{pmatrix} = 0$$

so that $\lim_{n\to\infty} |z_n| = \lim_{n\to\infty} (x_n^2 + y_n^2)^{\frac{1}{2}} = 0$

for the series to converge.

3) A series is <u>absolutely</u> <u>convergent</u> if

$$\sum_{n=1}^{\infty} |z_n| = \sum_n (x_n^2 + y_n^2)^{\frac{1}{2}}$$

converges. Absolute convergence is a sufficient condition for convergence since convergence of $\sum_n (x_n^2+y_n^2)^{1/2}$ implies

$$\sum_n |x_n| , \sum_n |y_n|$$

converge, which imply $\sum_n x_n$, $\sum_n y_n$ converge $\Rightarrow$ from Schwartz inequality

4) If a series $\sum_{n=0}^{\infty} a_n z^n$

converges when $z = z_1$, then it is absolutely convergent, for all z such that $|z| < |z_1|$ since

$$|a_n z_1^n| < M = \text{max. value of any single term}$$

$$|a_n z^n| = \left|\frac{z}{z_1}\right|^n |a_1 z_1^n| < M \left|\frac{z}{z_1}\right|^n$$

$$\Rightarrow \sum_{n=0}^{\infty} |a_n z^n| < M \sum_n \left|\frac{z}{z_1}\right|^n = \frac{M}{1 - \left|\frac{z}{z_1}\right|}$$

for $\left|\frac{z}{z_1}\right| < 1$

5) A power series is an analytic function at all points interior to its radius of convergence

6) If $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges

in a region $|z - z_0| < M$, then it must be the Taylor series expansion of $f(z)$ around z_0

$\Rightarrow$ series representations are unique

$\Rightarrow$ same for Laurent series

$$f(z_0) = a_0$$
$$f'(z_0) = \sum_{n=0}^{\infty} n a_n (z - z_0)^{n-1} \Big|_{z=z_0} = a_1$$

....

$$f^{(n)}(z_0) = n!\, a_n$$

Singularities and poles

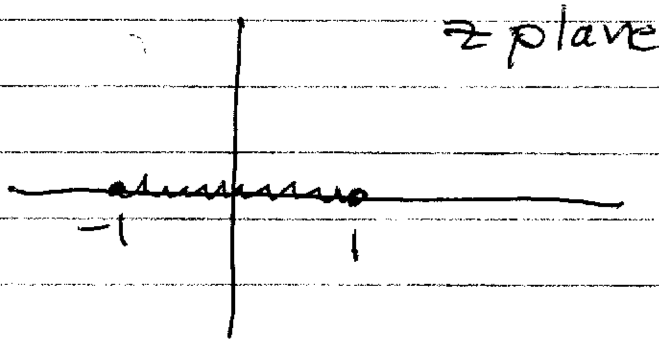
Consider a function $f(z)$ that is singular at a point z_0. If $f(z)$ is analytic everywhere in a neighbourhood of z_0, then z_0 is an isolated singular point of f.

example: $$f(z) = \frac{1}{z(z-1)^2}$$

$\Rightarrow$ two isolated singular points

example: $$f(z) = \frac{1}{(z^2-1)^{1/2}}$$

The points $z = \pm 1$ are not isolated singular points because a branch cut must be defined around $z = \pm 1$



In the vicinity of an isolated singular point z_0, $f(z)$ can be represented by a Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

with $a_n = \frac{1}{2\pi i} \oint_C dz' \frac{f(z')}{(z'-z_0)^{n+1}}$

If $a_{-N} \neq 0$ with $N > 0$ and $a_{-n} = 0$ for all $n > N$ then f has an Nth order pole at z_0.

example: $f(z) = \frac{1}{(z-1)^3}$ has a 3rd order pole at $z = 1$

$\Rightarrow$ when $N = 1$ $\Rightarrow$ simple pole

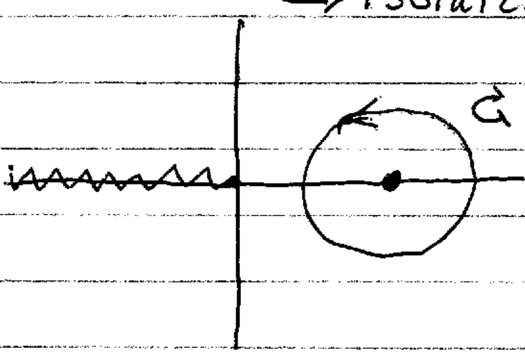
$\Rightarrow$ when $a_{-n} \neq 0$ as $n \to \infty$ $\Rightarrow$ essential singularity

example: $f(z) = \frac{\sin z}{z}$ $\Rightarrow$ no singularity at $z = 0$.

example: Let $f(z) = \ln(z) \frac{1}{(z-2)^2}$

Find Laurent series around $z = 2$

$\Rightarrow$ isolated singular point at $z = 2$



$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-2)^n$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{\ln(z)}{(z-2)^2} \frac{dz}{(z-2)^{n+1}}$$

Order of the pole at $z = 2$?

For $n=-2$,

$$a_{-2} = \frac{1}{2\pi i}\oint_C \ln(z)\,\frac{dz}{z-2} = \ln(2)$$

$$a_{-3} = \frac{1}{2\pi i}\oint_C \ln z = 0$$

$$a_{-4} = 0$$

Easier to expand $\ln(z)$ in Taylor series around $z=0$

$$\ln(z) = \ln(2) + \frac{1}{1!}(\ln z)'\Big|_2 (z-2) + \frac{1}{2!}(\ln z)''\Big|_2 (z-2)^2 + \cdots$$

$$f(z) = \frac{1}{(z-2)^2}\Big[\ln(2) + (\;)(z-2) + (\;)(z-2)^2 \cdots\Big]$$

example: $f(z) = e^{\frac{1}{z}}$ has essential sing. at $z=0$.

$\Rightarrow$ Laurent series by integration messy. Let $s=\frac{1}{z}$

$$e^s = \sum_{n=0}^{\infty} \frac{1}{n!} s^n \quad \text{for } |s|<\infty$$

$$f(z) = e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{1}{z^n} \quad |z|>0$$

$\Rightarrow$ must be Laurent series