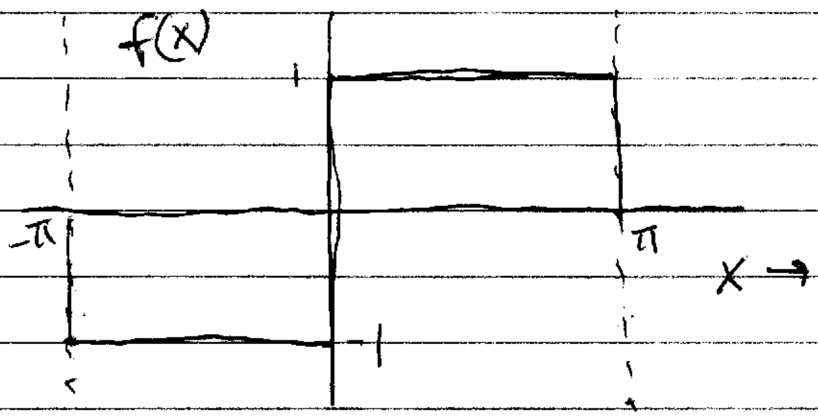
Using basis functions

example: square-wave function

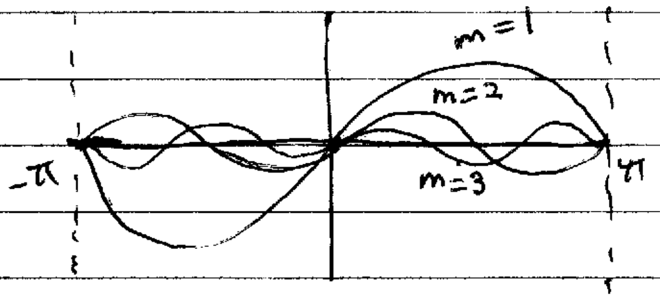


$\Rightarrow$ periodic over 2π

$$f(x) = \sum_{m=1}^{\infty} C_s^m \sin mx + \sum_{m=0}^{\infty} C_c^m \cos mx$$

$f(x)$ is odd around $x=0$ so $C_c = 0$.

$$f(x) = \sum_{m=1}^{\infty} C_s^m \sin(mx)$$



$f(x)$ is even around $x = \frac{\pi}{2}$

$\Rightarrow$ Keep odd m

$$f(x) = \sum_{m\ \text{odd}} C_s^m \sin(mx)$$

Multiply by $\sin(nx)$ and integrate $(-\pi, \pi)$

$$\int_{-\pi}^{\pi} dx\, f(x) \sin nx = \sum_m C_s^m \delta_{mn}\, 2\pi\, \frac{1}{2} = \pi C_s^n$$

$$\langle \sin^2(nx) \rangle = \frac{\langle 1 - \cos 2nx \rangle}{2} = \frac{1}{2}$$

$$\pi C_s^n = -\int_{-\pi}^{0} dx \sin nx + \int_{0}^{\pi} dx \sin nx$$

$$= \frac{\cos nx}{n}\Big|_{-\pi}^{0} - \frac{\cos nx}{n}\Big|_{0}^{\pi} = \frac{1}{n}\left[1 - (-1)^n - (-1)^n + 1\right]$$

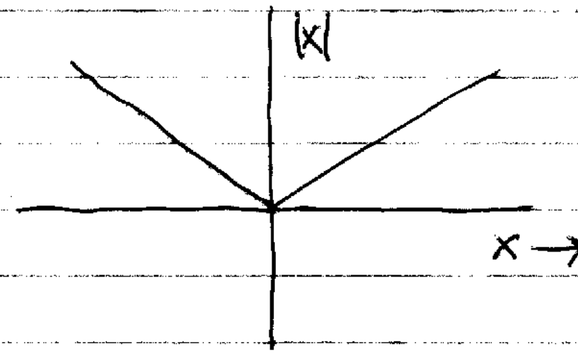
$= 0$ for n even, $\frac{4}{n}$ for n odd

$$f(x) = \frac{4}{\pi} \sum_{m \text{ odd.}} \frac{1}{m} \sin mx$$

<u>example</u> $f(x) = |x|$ over $(-1, 1)$

$\Rightarrow$ use Legendre polynomials

$$|x| = \sum_{n=0}^{\infty} c_n P_n(x)$$



Since $|x|$ is even, need only ~~B~~ n even

$$\int_{-1}^{1} dx\, |x|\, P_m(x) = c_m \frac{2}{2m+1}$$

$$c_m = \frac{2m+1}{2} \int_{-1}^{1} dx\, |x|\, P_m(x) = (2m+1) \int_{0}^{1} dx\, x\, P_m(x)$$

for m even

$= 0$ for m odd

$$c_m = (2m+1) \int_0^1 dx\, x \frac{1}{m!\, 2^m} \frac{d^m}{dx^m} (x^2-1)^m$$

$$= -\frac{(2m+1)}{m!\, 2^m} \int_0^1 dx \frac{d^{m-1}}{dx^{m-1}} (x^2-1)^m$$

$$= -\frac{2m+1}{m!\, 2^m} \frac{d^{m-2}}{dx^{m-2}} (x^2-1)^m \Big|_0^1$$

Since $(x^2-1)^m = (x-1)^m (x+1)^m$

$\Rightarrow$ taking $m-2$ derivatives leaves powers of $(x-1)^p$ with $p \geq 2$

$\Rightarrow$ value at $x=1$ is zero

$$C_m = \frac{2m+1}{m!\,2^m} \frac{d^{m-2}}{dx^{m-2}} (x^2-1)^m \Big|_0$$

Binomial expansion

$$(x^2-1)^m = \sum_{l=0}^{m} \frac{m!}{l!(m-l)!} x^{2l} (-1)^{m-l}$$

Only surviving term with $x=0$ is

$$m-2 = 2l \Rightarrow l = \frac{m-2}{2} = \frac{m}{2} - 1$$

$$C_m = \frac{2m+1}{m!\,2^m} \frac{m!}{\left(\frac{m}{2}-1\right)!\left(m-\frac{m}{2}+1\right)!} (m-2)!(-1)^{\frac{m}{2}+1}$$

$$C_m = \frac{(2m+1)}{2^m} \frac{(m-2)!(-1)^{\frac{m}{2}+1}}{\left(\frac{m}{2}+1\right)!\left(\frac{m}{2}-1\right)!} \quad m \text{ even}$$

$= 0 \quad m$ odd

Representing Greens functions with eigen functions

$$\mathcal{L}\, G(x,x') + \lambda w\, G(x,x') = \delta(x-x')$$

where λ is a fixed value but not an eigen value

$$G(x,x') = \sum_m c_m(x')\, \varphi_m(x)$$

$\Rightarrow$ remember that $\mathcal{L}$ is an operator in x.

Since the $\varphi_m(x)$ are basis functions,

$$\mathcal{L}\, \varphi_m(x) + \lambda_m w\, \varphi_m = 0$$

~~Inserted~~ Substituting $G(x,x')$ into the diff. eqn yields

$$\sum_m w\, c_m(\lambda - \lambda_m)\, \varphi_m(x) = \delta(x-x')$$

Multiply by $\varphi_n^*(x)$ and integrate (a,b)

$\Rightarrow$ using orthogonality

$$\sum_m c_m(\lambda-\lambda_m) \underbrace{\int_a^b dx\, \varphi_n^*(x)\, \varphi_m(x)\, w}_{\delta_{mn}} = \varphi_n^*(x')$$

$$c_n = \frac{\varphi_n^*(x')}{\lambda - \lambda_n} \qquad G(x,x') = \sum_m \frac{\varphi_m^*(x')\, \varphi_m(x)}{\lambda - \lambda_m}$$

$\Rightarrow$ no Greens function for λ an eigen value

Wave equation in spherical coordinates

We previously solved wave equations in cartesian coordinates in 1-D by carrying out Fourier transforms or using sin() or cos() series. What about cylindrical or spherical systems?

Consider a spherically symmetric wave in 3-D

$$\frac{\partial^2}{\partial t^2} y - c^2 \nabla^2 y = 0$$

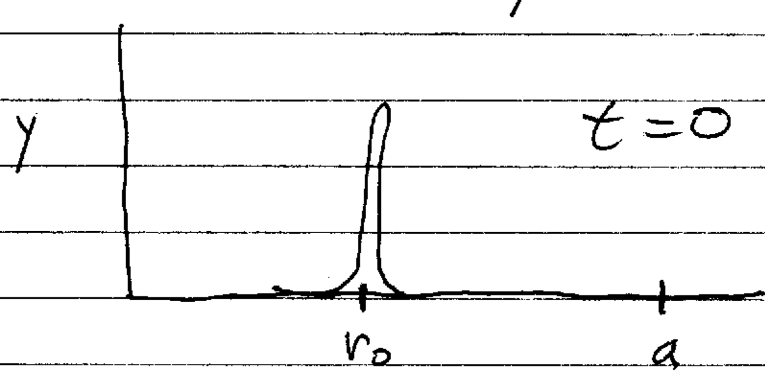
$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}$$

with initial conditions

$$y(r, t=0) = \delta(r - r_0)$$

$$\dot{y}(r, t=0) = 0$$

and BCs $y(a, t) = 0$



Want a set of basis functions for this system. Let

$$y(r,t) = \sum_{m=1}^{\infty} c_m(t)\,\varphi_m(r)$$

$$\sum_m \ddot{c}_m(t)\,\varphi_m(r) - c^2 \sum_m c_m \nabla^2 \varphi_m(r) = 0$$

Choose

$$\nabla^2 \varphi_m(r) + k_m^2\,\varphi_m(r) = 0$$

$$\Rightarrow \sum_m \left(\ddot{c}_m(t) + k_m^2 c^2 c_m\right)\varphi_m = 0$$

Satisfied for all r if

$$\ddot{c}_m + k_m^2 c^2 c_m = 0$$

$$\Rightarrow c_m \sim \cos(k_m c t),\ \sin(k_m c t)$$

$\Rightarrow$ keep $\cos()$ since $\dot{y} \sim \dot{c}_m = 0$ at $t=0$

$$\Rightarrow c_m(t) = c_m(0)\cos(k_m c t)$$

$$y(r,t) = \sum_m c_m(0)\cos(k_m c t)\,\varphi_m(r)$$

$\Rightarrow \varphi_m(r)$ still unknown

$$\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} \mathcal{R}_m(r) + k_m^2 r^2 \mathcal{R}_m(r) = 0$$

$\Rightarrow$ this is of Stürm-Liouville form

$$r^2 \mathcal{R}_m'' + 2r\mathcal{R}_m' + k_m^2 r^2 \mathcal{R}_m = 0$$

Bessel's eqn is given by

$$r^2 g'' + r g' + (k^2 r^2 - \nu^2) g = 0$$

Let $\mathcal{R}_m = \dfrac{Q_m}{r^{1/2}}$

$$r^2 \left[\frac{Q_m''}{r^{1/2}} - 2 \frac{Q_m' \frac{1}{2}}{r^{3/2}} + \frac{1}{2}\frac{3}{2} \frac{Q_m}{r^{5/2}} \right]$$

$$+ 2r \left[\frac{Q_m'}{r^{1/2}} - \frac{1}{2} \frac{Q_m}{r^{3/2}} \right] + k_m^2 r^2 \frac{Q_m}{r^{1/2}} = 0$$

$$r^2 Q_m'' + r Q_m' + \left(\frac{3}{4} - 1\right) Q_m + k_m^2 r^2 Q_m = 0$$

$$r^2 Q_m'' + r Q_m' + \left(k_m^2 r^2 - \frac{1}{4}\right) Q_m = 0$$

$\Rightarrow$ Bessel's eqn with $\nu = \frac{1}{2}$ and $k = k_m$

$\Rightarrow$ solutions are $J_{\frac{1}{2}}(k_m r)$, $Y_{\frac{1}{2}}(k_m r)$

Behavior of Bessel solutions near $r=0$

$\Rightarrow$ RSP

$$r^2 g'' + r g' - \nu^2 g = 0$$

$$g \sim r^p$$

$$p(p-1) + p - \nu^2 = 0$$

$$p = \pm\nu = \pm\tfrac{1}{2}$$

$$J_{\frac{1}{2}}(k_m r) \sim r^{1/2}, \quad Y_{\frac{1}{2}} \sim r^{-1/2}$$

The required S-L BCs at $r=0$

$$r^2 \mathcal{C}_m \mathcal{C}_n' \Big|_{r=0} = 0$$

$\Rightarrow Y_{\frac{1}{2}}$ does not satisfy this condition

$$r^2 \frac{1}{r^{1/2}} \frac{1}{r^{3/2}} \neq 0 \text{ at } r=0$$

Thus, $\quad \mathcal{C}_m = \dfrac{1}{r^{1/2}} J_{\frac{1}{2}}(k_m r)$

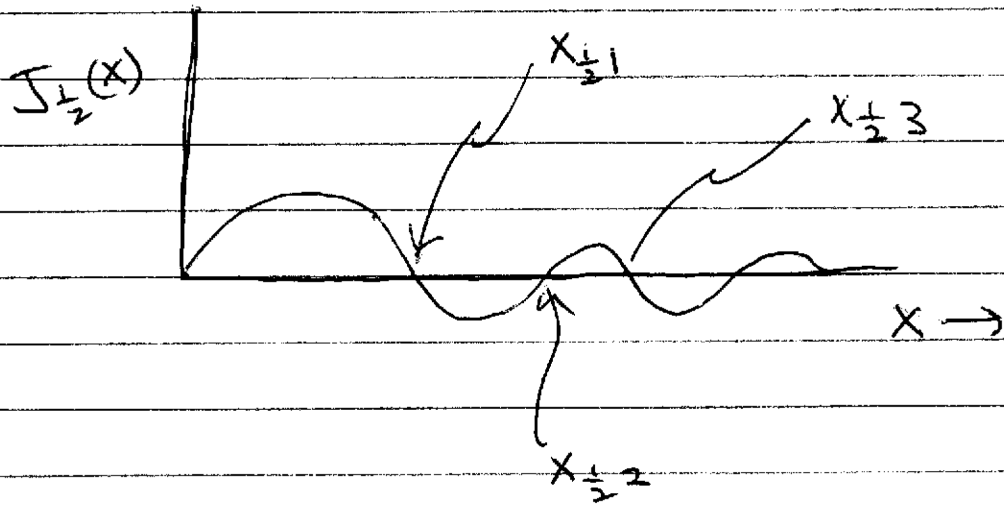
BC at $r=0$ ok since $\mathcal{C}_m \sim$ const

$$r^2 \mathcal{C}_m \mathcal{C}_m' \Big|_{r=0} = 0$$

At $r=a$ require

$$\varphi_m(r=a) = 0 = \frac{1}{a^{1/2}} J_{\frac{1}{2}}(k_m a) = 0$$

$$\Rightarrow J_{\frac{1}{2}}(k_m a) = 0$$



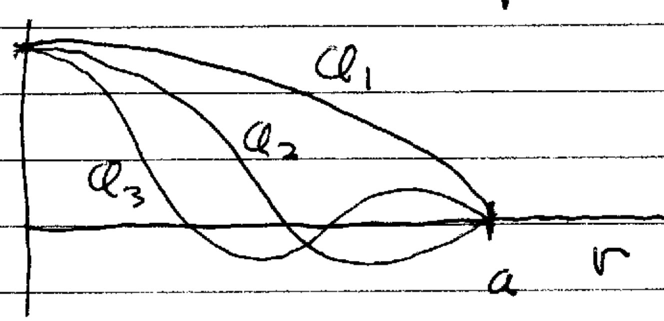
As x increases $J_{\frac{1}{2}}(x)$ is oscillatory.
why?

$x_{\nu m}$ is the mth zero of a Bessel function of order ν $\Rightarrow$ these are tabulated

Thus $\quad k_m a = x_{\frac{1}{2}m}$

$$k_m = \frac{x_{\frac{1}{2}m}}{a} \quad \text{is the eigenvalue}$$

Thus, $\quad \varphi_m(r) = \dfrac{1}{r^{1/2}} J_{\frac{1}{2}}\left(\dfrac{x_{\frac{1}{2}m} r}{a}\right)$



Are they orthogonal?

Normalization: $\omega(r) = r^2$

$$\int_0^a dr\, r^2\, \mathcal{Q}_m^2(r) = \int_0^a dr\, r\, J_{\frac{1}{2}}^2\left(k_m r\right)$$

Let $s = \frac{r}{a}$ $\quad k_m = \frac{x_{\frac{1}{2}m}}{a}$

$$= a^2 \int_0^1 ds\, s\, J_{\frac{1}{2}}^2\left(x_{\frac{1}{2}m} s\right)$$

$$= \frac{a^2}{2} J_{\frac{3}{2}}^2\left(x_{\frac{1}{2}m}\right) \equiv N_m^2$$

Initial conditions

$$y(r,0) = \delta(r - r_0) = \sum_m c_m(0)\, \mathcal{Q}_m(r)$$

Multiply by $r^2 \mathcal{Q}_n^*(r)$ and integrate

$$c_n(0)\, N_n^2 = \int_0^a dr\, r^2\, \mathcal{Q}_n(r)\, \delta(r - r_0)$$

$$= r_0^2\, \mathcal{Q}_n(r_0)$$

$$c_n(0) = \frac{r_0^2\, \mathcal{Q}_n(r_0)}{N_n^2}$$

$$Y(r,t) = \sum_{m=1}^{\infty} \frac{r_0^2}{\frac{a^2}{2} J_{\frac{3}{2}}^2\left(x_{\frac{1}{2}m}\right)} \frac{1}{r_0^{1/2}} J_{\frac{1}{2}}\left(x_{\frac{1}{2}m} \frac{r_0}{a}\right)$$

$$\otimes\ \frac{J_{\frac{1}{2}}\left(x_{\frac{1}{2}m} r/a\right)}{r^{1/2}} \cos\left(\frac{x_{\frac{1}{2}m} c t}{a}\right)$$

Each $a_m(r)$ corresponds to a standing wave in spherical geometry

Each wave has its own characteristic frequency

$$\omega_m = \frac{x_{\frac{1}{2}m}\, c}{a}$$

Each oscillates independently

$\Rightarrow$ You generally want to choose your basis functions to match the differential operator governing the system.

Laplace's eqn in a cylindrical system

Laplace's eqn emerges when solving for the electrostatic potential in a system with no discrete charges.

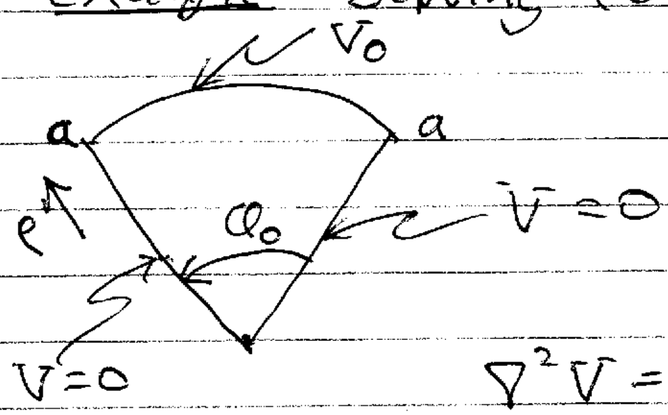
$$\nabla \cdot \underset{\sim}{E} = 4\pi \rho = 0$$

$$\nabla \times \underset{\sim}{E} = 0 \Rightarrow \underset{\sim}{E} = -\nabla V$$

$$\Rightarrow \nabla^2 V = 0$$

with V typically specified on conducting boundaries.

example Solving for V in a wedge



Use a cylindrical coordinate system ρ, φ

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho}V + \frac{1}{\rho^2}\frac{\partial^2}{\partial \varphi^2}V = 0$$

Choose

$$V = \sum_m c_m \Phi(\varphi) R_m(\rho)$$

$$\nabla^2 V = \sum_m c_m \left[\Phi_m(\varphi)\frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho}R_m(\rho) + \frac{R_m(\rho)}{\rho^2}\frac{\partial^2}{\partial \varphi^2}\Phi_m \right]$$

$$= \sum_m c_m \frac{\Phi_m(\varphi) R_m(\rho)}{\rho^2}\left[\frac{1}{R_m}\rho\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho}R_m + \frac{1}{\Phi_m}\frac{\partial^2}{\partial \varphi^2}\Phi_m \right]$$

Since $\nabla^2 V$ must be zero for all ρ, φ must have

$$\underbrace{\frac{1}{R_m}\rho\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}R_m}_{\text{function of } \rho} + \underbrace{\frac{1}{\Phi_m}\frac{\partial^2}{\partial\varphi^2}\Phi_m}_{\text{function of } \varphi} = 0$$

$\Rightarrow$ must each be constant

$$\Rightarrow \quad \frac{1}{\Phi_m}\frac{\partial^2}{\partial\varphi^2}\Phi_m = -\nu_m^2$$

$$\frac{\partial^2}{\partial\varphi^2}\Phi_m + \nu_m^2\Phi_m = 0$$

$$\Phi_m \sim \sin\nu_m\varphi \, , \; \cos\nu_m\varphi$$

Choose $\sin\nu_m\varphi$ since $V=0$ at $\varphi=0$

Require $\sin\nu_m\varphi_0 = 0$ so $V=0$ at $\varphi=\varphi_0$

$$\Rightarrow \nu_m\varphi_0 = m\pi$$

$$V = \sum_{m=1}^{\infty} c_m \sin\left(\frac{m\pi}{\varphi_0}\varphi\right) R_m(\rho)$$

Also have

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}R_m - \frac{\nu_m^2}{\rho^2}R_m = 0$$

$$\rho \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} R_m - \nu_m^2 R_m = 0$$

$$\rho^2 \frac{\partial^2}{\partial \rho^2} R_m + \rho \frac{\partial}{\partial \rho} R_m - \nu_m^2 R_m = 0$$

$\Rightarrow$ Euler eqn $\quad R_m \sim \rho^{\gamma}$

$$\gamma(\gamma - 1) + \gamma - \nu_m^2 = 0$$

$$\gamma^2 = \nu_m^2 \quad \Rightarrow \quad \gamma = \pm \nu_m$$

$$R_m \sim \rho^{\pm \nu_m}$$

$\Rightarrow$ require R_m remain bounded at $\rho = 0$

$$R_m \sim \rho^{\nu_m} \sim \rho^{\frac{m\pi}{\alpha_0}}$$

$$V = \sum_{m=1}^{\infty} C_m \sin \nu_m \alpha \, \rho^{\nu_m}$$

$$\nu_m = \frac{m\pi}{\alpha_0}$$

Determine C_m by matching the solution at $\rho = a$ where $V = V_0$

$$V_0 = \sum_{m=1}^{\infty} C_m \sin \nu_m \alpha \; a^{\nu_m}$$

To solve for C_m must eliminate the sum over m

$\Longrightarrow$ multiply by $\sin(\nu_n \varphi)$ and integrate $(0, \varphi_0)$

$$V_0 \int_0^{\varphi_0} d\varphi \, \sin \nu_n \varphi = \sum_m a^{\nu_m} \int_0^{\varphi_0} d\varphi \, C_m \sin \nu_m \varphi \, \sin \nu_n \varphi$$

$\Longrightarrow$ $\sin \nu_m \varphi$, $\sin \nu_n \varphi$ are orthogonal unless $m = n$.

why?

$$-V_0 \left. \frac{\cos \nu_n \varphi}{\nu_n} \right|_0^{\varphi_0} = \sum_m a^{\nu_m} \tfrac{1}{2} \varphi_0 \, \delta_{mn} C_m$$

$$= a^{\nu_n} \tfrac{1}{2} \varphi_0 C_n$$

$$-\frac{V_0}{\nu_n} \Big[\underbrace{\cos(\underbrace{n\pi}_{}) - 1}_{\substack{=0 \ n \text{ even} \\ =-2 \ n \text{ odd}}} \Big] = a^{\nu_n} \tfrac{1}{2} \varphi_0 C_n$$

$(-1)^n$

$$C_n = \frac{4 V_0}{\varphi_0} \frac{1}{a^{\nu_n}} \frac{1}{\nu_n}$$

$$V = \frac{4 V_0}{\varphi_0} \sum_{\substack{m=1 \\ \text{odd}}}^{\infty} \frac{1}{\nu_m} \sin \nu_m \varphi \left(\frac{\ell}{a} \right)^{\nu_m}$$

$$\nu_m = \frac{m\pi}{\varphi_0}$$

Note the the functions in φ are oscillatory while those in ρ are not.

$\Rightarrow$ because of Laplace's eqn

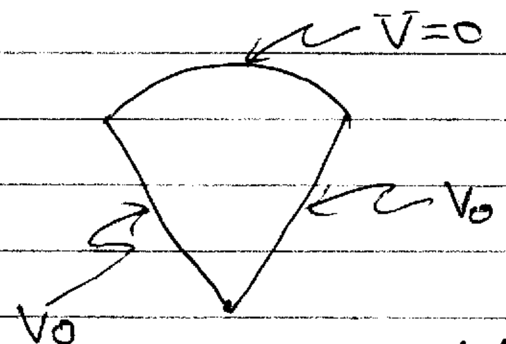
$$\underbrace{\frac{1}{R_m}\rho\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}R_m}_{\nu_m^2} + \underbrace{\frac{1}{\Phi_m}\frac{\partial^2}{\partial\varphi^2}\Phi_m}_{-\nu_m^2} = 0$$

not oscillatory   oscillatory

$\Rightarrow$ the R_m are not basis functions in the Stürm-Liouville sense

$\Rightarrow$ the Φ_m are basis functions

$\Rightarrow$ Always choose oscillatory functions along the boundary where a nonzero V is specified



$\Rightarrow$ need oscillatory functions in ρ

What about ?