Inhomogeneous equations

Consider an nth order equation

$$\mathcal{L}(y) = y^{(n)} + a_{n-1}(x)\, y^{(n-1)} + \cdots a_0 y^{(0)} = b(x)$$

The general solution to $\mathcal{L}(y) = b$ is the particular solution y_p:

$$\mathcal{L}(y_p) = b(x)$$

plus any linear combination of homogeneous solutions

$$y = y_p + \sum_{m=1}^{n} c_m \phi_m(x)$$

$$\mathcal{L}(y) = \mathcal{L}(y_p) + \sum_{m=1}^{\infty} c_m \underbrace{\mathcal{L}(\phi_m)}_{0}$$

$$= b(x)$$

Thus, an nth order inhomogeneous eqn. has n+1 possible solutions.

example: First order eqn

$$y' + a_0(x)\, y = b(x)$$

$\Rightarrow$ can always solve this eqn.

$\Rightarrow$ The integrating factor is

$$e^{\int_{x_0}^{x} dx' a_0(x')}$$

$$e^{\int_{x_0}^{x} dx\, a_0(x')}\left(y' + a_0(x)y\right) = b\, e^{\int_{x}^{x} dx' a_0(x')}$$

$$\frac{d}{dx}\left[y\, e^{\int_{x_0}^{x} dx' a_0(x')}\right] = b\, e^{\int_{x_0}^{x} dx' a_0(x')}$$

$$y\, e^{(\;)} = \int_{x_0}^{x} dx''\, b(x'')\, e^{\int_{x_0}^{x''} dx' a_0(x')} + c_1$$

$$y = e^{-\int_{x_0}^{x} dx' a_0(x')} \int_{x_0}^{x'} dx''\, b(x'')\, e^{\int_{x_0}^{x''} dx' a_0(x')}$$

$$+ c_1 e^{-\int_{x_0}^{x} dx' a_0(x')}$$

particular solution   homogeneous solution

The difference between solving homogeneous in inhomogeneous eqns. is associated with finding the particular solution.

$\Rightarrow$ explain how to do this

Method of undetermined coefficients

This works if have constant coefficients and with $b(x)$ a combination of

$$e^{px},\ \sin^{q}(x),\ \cos^{q}(x)$$

or Eulers eqn with $b(x)$ a polynomial

example:

$$y'' - 3y' + 2y = e^{4x}$$

Guess that $y_p = g e^{4x}$

$$g e^{4x}\left(16 - 3(4) + 2\right) = e^{4x}$$

$$g = \frac{1}{6} \implies y_p = \frac{1}{6} e^{4x}$$

$\Rightarrow$ the operator on the LHS does not alter the functional form of the solution.

example:

$$y'' + y = e^{2x} \sin x$$

Let $$y_p = g e^{2x} \sin x + h e^{2x} \cos x$$

$$y_p'' = g\left[2(2)e^{2x}\sin x + 2(2)e^{2x}\cos x - e^{2x}\sin x\right]$$
$$+ h\left[2(2)e^{2x}\cos x - 2(2)e^{2x}\sin x - e^{2x}\cos x\right]$$
$$= e^{2x}\sin x\left[4g - g - 4h\right]$$
$$+ e^{2x}\cos x\left[4g + 4h - h\right]$$

$$y_p'' + y_p = e^{2x}\sin x\left[3g - 4h + g\right]$$
$$+ e^{2x}\cos x\left[4g + 3h + h\right] = e^{2x}\sin x$$

$\Rightarrow$ equation functional forms

$$4(g-h) = 1$$
$$4(g+h) = 0 \quad \Rightarrow g = -h$$
$$g = \frac{1}{8}, \quad h = -\frac{1}{8}$$

$$y = \frac{1}{8}e^{2x}\sin x - \frac{1}{8}e^{2x}\cos x + c_1 \sin x$$
$$+ c_2 \cos x$$

example: $\ddot{y} + y = \sin \omega t$

$\Rightarrow$ resonance cavity driven by a source

Let $y = A \sin \omega t$

$$(-\omega^2 + 1) A = 1 \implies A = \frac{1}{1-\omega^2}$$

$$y = \frac{1}{1-\omega^2} \sin \omega t + c_1 \sin t + c_2 \cos t$$

What if $\omega = 1$?

$$\ddot{y} + y = \sin t$$

$y = At \cos t \implies$ growing solution since resonance

$$\left[\cancel{At(-)\cos t} - 2A \sin t + \cancel{At \cos t} \right] = \sin t$$

$$-2A = 1 \implies A = -\frac{1}{2}$$

$$y = -\frac{1}{2} t \cos t + c_1 \sin t + c_2 \cos t$$

Example $\quad x^2 y'' + x y' + 2y = 1$

Try $y_p = A$

$$2A = 1 \implies A = \frac{1}{2}$$

$y_p = \frac{1}{2}$ + homogeneous

homogeneous $\Rightarrow$ powerlaw

$$\sim y_h^p$$

$$p(p-1)+p+2=0$$

$$p^2=-2 \Rightarrow p=\pm i\sqrt{2}$$

$$y=\frac{1}{2}+c_1 x^{i\sqrt{2}}+c_2 x^{-i\sqrt{2}}$$

Uniqueness of particular solutions

Are particular solutions unique?

Consider $\quad \mathcal{L}(y_{p1})=6$

$$\mathcal{L}(y_{p2})=6$$

Let $\epsilon = y_{p1}-y_{p2}$

$$\mathcal{L}(\epsilon)=6-6=0$$

$$\mathcal{L}(\epsilon)=0$$

$\Rightarrow$ ϵ can be any homogeneous solution

$\Rightarrow$ particular solutions not unique

Green's Functions

Consider the equation

$$\mathcal{L}(y) = b(x)$$

We define the Green's function $G(x,x')$ to be the solution of

$$\mathcal{L}\, G(x,x') = \delta(x-x')$$

$\Rightarrow$ note that $\mathcal{L}$ is an operator in x.

In terms of the Green's function can write y_p as

$$y_p = \int_{-\infty}^{\infty} dx'\, b(x')\, G(x,x')$$

$$\mathcal{L}(y_p) = \int_{-\infty}^{\infty} dx'\, b(x')\, \mathcal{L}\, G(x,x')$$
$$= \int_{-\infty}^{\infty} dx'\, b(x')\, \delta(x-x') = b(x)$$

$\Rightarrow$ The solution of an inhomogeneous equation reduces to solving for

$$G(x,x')$$

Consider the general second order eqn.

$$\frac{d^2}{dx^2} G + a_1 \frac{dG}{dx} + a_0 G = \delta(x - x')$$

Suppose φ_1 and φ_2 are the homogeneous solutions. Then for $x \neq x'$

$$G = c_1^+ \varphi_1(x) + c_2^+ \varphi_2(x) \qquad x > x'$$

$$G = c_1^- \varphi_1(x) + c_2^- \varphi_2(x) \qquad x < x'$$

$\Rightarrow$ note that $c_1^+ \neq c_1^-$, $c_2^+ \neq c_2^-$

since G must have a jump at $x = x'$.

Typically, we will have boundary conditions on y. Can use the BC's to determine $c_1^\pm$ and $c_2^\pm$.

Suppose we are solving for y over the interval (a, b) and want $y(a) = y(b) = 0$

$$y(a) = \int_a^b dx'\, b(x')\, G(a, x') = 0$$

$\Rightarrow$ require $G(a, x') = 0$ for $a < x'$

Similarly

$$y(b) = \int_a^b dx'\, b(x')\, G(b,x') = 0$$

$$\Rightarrow G(b,x') = 0 \text{ for } b > x'$$

Can always choose a combination of $\mathcal{Q}_1, \mathcal{Q}_2$ to go to zero at the boundaries

Let $\mathcal{Q}_-(x)$ be a homogeneous solution that is zero at $x = a$.

Let $\mathcal{Q}_+(x)$ be a homogeneous solution at is zero at $x = b$.

Then $\quad G = C_+ \mathcal{Q}_+(x) \quad x > x'$

$\quad\quad = C_- \mathcal{Q}_-(x) \quad x < x'$

$\Rightarrow G = 0$ at $x = a, b$

$\Rightarrow$ boundaries could be at $\pm\infty$
so $\mathcal{Q}_+ \to 0$ as $x \to \infty$
and
$\mathcal{Q}_- \to 0$ as $x \to -\infty$

Need to determine C_+ and C_- by ~~matching~~ examining the solutions around $x = x'$.

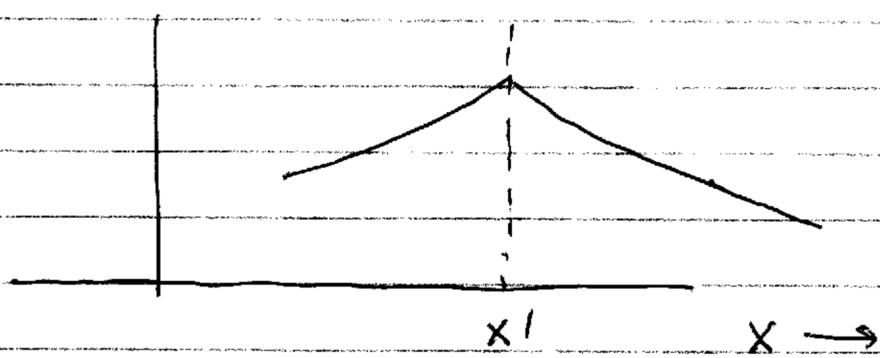
$\Rightarrow$ the dominant term on the LHS of the diff. eqn for G near $x = x'$ is d^2G/dx^2 so

$$\frac{d^2 G}{dx^2} = \delta(x - x')$$

Integrating across $x = x'$

$$\left.\frac{dG}{dx}\right|_{x = x' - \epsilon}^{x = x' + \epsilon} = 1$$

Thus, the slope of G suffers a jump at $x = x'$.



Note that dG/dx and G remain finite at $x = x'$

$\Rightarrow$ only $\dfrac{d^2G}{dx^2}$ is singular

Because dG/dx is finite, G is continuous at $x = x'$.

$$G \Big|_{x=x'-\epsilon}^{x=x'+\epsilon} = 0$$

$\Rightarrow$ these results are a consequence of the fact that the integral of $\delta(x-x')$ is bounded so

$$\frac{dG}{dx} = \int^{x} dx'' \, \delta(x''-x')$$
$$= H(x-x') = \begin{cases} 1 & x > x' \\ 0 & x < x' \end{cases}$$

Continuity of $G(x,x')$ at $x = x'$ requires

$$c_+ \varphi_+(x') = c_- \varphi_-(x')$$

Let $c_+ = \varphi_-(x')\, c$

$$c_- = \varphi_+(x')\, c$$

so

$$G = \begin{cases} c\, \varphi_+(x)\, \varphi_-(x') & x > x' \\ c\, \varphi_-(x)\, \varphi_+(x') & x < x' \end{cases}$$

The jump in the slope of G yields c

$$\left.\frac{\partial G}{\partial x}\right|_{x'-\epsilon}^{x'+\epsilon} = 1 = c\left[\varphi_+'(x')\varphi_-(x') - \varphi_-'(x')\varphi_+(x')\right]$$

$$= c\left[\overline{W}(\varphi_-(x'), \varphi_+(x'))\right]$$

Thus,

$$G = \frac{\varphi_+(x)\,\varphi_-(x')}{\overline{W}(\varphi_-(x'), \varphi_+(x'))} \qquad x > x'$$

$$= \frac{\varphi_+(x')\,\varphi_-(x)}{\overline{W}[\varphi_-(x'), \varphi_+(x')]} \qquad x < x'$$

Can add a homogeneous solution to the particular solution given by the integral over the Green's function?

$\Rightarrow$ only if it satisfies both BC's at a, b

$\Rightarrow$ does not typically satisfy the BCs.

example $\quad y'' - y = \delta(x)$

where $y \to 0$ as $|x| \to \infty$.

$$\frac{\partial^2}{\partial x^2} G - G = \delta(x - x')$$

For $x \neq x'$,

$$\frac{\partial^2 G}{\partial x^2} - G = 0$$

$$\Rightarrow G \sim e^{\pm x}$$

For $x > x'$, keep bounded solution as $x \to \infty$

$$G = c_+ e^{-(x-x')}$$

For $x < x'$, keep bounded solution as $x \to -\infty$

$$G = c_- e^{(x-x')}$$

Continuity of $G \Rightarrow c_+ = c_- = c$

Jump in slope of $G \Rightarrow c$

$$\left.\frac{dG}{dx}\right|_{x'-\epsilon}^{x'+\epsilon} = 1 = -c - c = -2c$$

$$c = -\frac{1}{2}$$

$$G = -\tfrac{1}{2} e^{-(x-x')} \qquad x > x'$$

$$= -\tfrac{1}{2} e^{(x-x')} \qquad x < x'$$

Solution for y,

$$y = -\tfrac{1}{2} \int_{-\infty}^{\infty} dx' \, b(x') \, e^{-|x-x'|}$$

$\Rightarrow$ can add homogeneous solution?

Consider the same equation but over the interval $(0,1)$ with $y(0) = y(1) = 0$

For $x > x'$, want to choose the homogeneous solution that vanishes at 1,

$$G(x,x') = C_+ \sinh(x-1) \qquad x > x'$$

For $x < x'$, ~~[illegible]~~ want solution that vanishes at 0,

$$G(x,x') = C_- \sinh(x) \qquad x < x'$$

Continuity at $x = x'$ requires

$$C_+ = C \sinh(x')$$

$$C_- = C \sinh(x' - 1)$$

$$G(x,x') = c\,\sinh(x')\sinh(x-1) \qquad x > x'$$

$$= c\cdot\sinh(x)\sinh(x'-1) \qquad x < x'$$

Jump condition

$$c\left[\cosh(x'-1)\sinh(x') - \cosh(x')\sinh(x'-1)\right]$$

$$\sinh(a \pm b) = \sinh(a)\cosh(b) \pm \cosh(a)\sinh(b)$$

$$b = x'-1$$

$$a = x'$$

$$c\left[\sinh\left[x' - (x'-1)\right]\right] = c\sinh(1) = 1$$

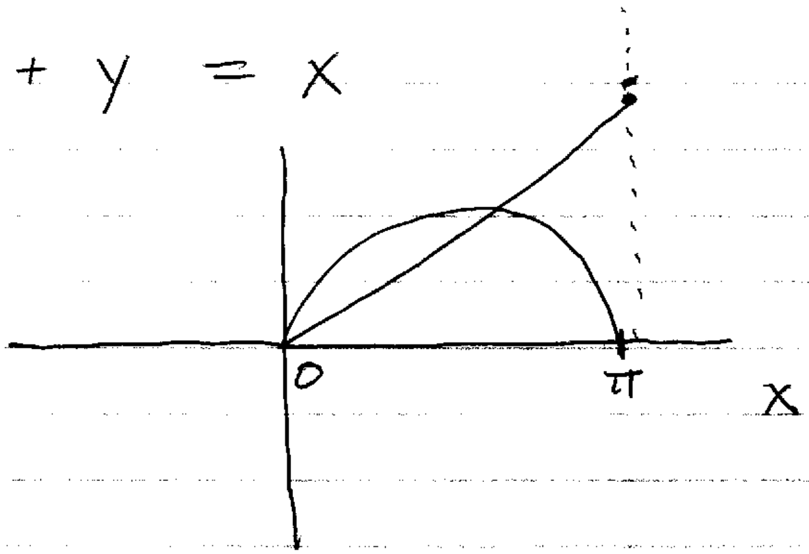
$$c = \frac{1}{\sinh(1)}$$

$$G = \frac{\sinh(x-1)\sinh(x')}{\sinh(1)} \qquad x > x'$$

$$= \quad x \longleftrightarrow x' \qquad x < x'$$

example $y'' + y = x$

$y(0) = y(\pi) = 0$



$$\frac{\partial^2 G}{\partial x^2} + G(x, x') = \delta(x - x')$$

For $x \neq x'$

$G \sim \sin x, \cos x$

For $x < x'$:

$G = C_- \sin x$

For $x > x'$:

$G = C_+ \sin x$

Continuity for $C_+ = C_-$ but no jump in slope

$\Rightarrow$ no Green's function

Proof that this problem has no solution.

$\Rightarrow$ multiply equ by $\sin x$ and integrate 0 to π.

$$\int_0^\pi dx \, \sin x \, (y'' + y) = \int_0^\pi dx \; x \sin x$$

$$\cancel{\sin x \, y' \Big|_0^\pi} - \int_0^\pi dx \, y' \cos x + \int_0^\pi dx \, (\sin x) \, y$$

$$= - \cancel{y \cos x \Big|_0^\pi} - \int_0^\pi dx \, y \sin x + \int_0^\pi dx \, y \sin x$$

$$= 0 = \int_0^\pi dx \; x \sin x \neq 0$$

$\Rightarrow$ contradiction

$\Rightarrow$ Not all inhomogeneous boundary value problems have solutions

When the homogeneous solution satisfies both BCs, have a resonance. The source must then have equal positive and negative contributions when weighted over the resonant solution so that the solution will remain bounded.

Non-homogeneous Boundary conditions

Consider the general second order equation with

$$y(a) = y_a, \quad y(b) = y_b$$

Let

$$y_p = \int_a^b dx' \, G(x,x') \, b(x')$$

where $G(a,x') = G(b,x') = 0$

$$\Longrightarrow y_p(a) = y_p(b) = 0$$

$$y = y_p + c_1 \mathcal{U}_1(x) + c_2 \mathcal{U}_2(x)$$

$$y(a) = c_1 \mathcal{U}_1(a) + c_2 \mathcal{U}_2(a)$$

$$y(b) = c_1 \mathcal{U}_1(b) + c_2 \mathcal{U}_2(b)$$

c_1, c_2 will have solutions if

$$\begin{vmatrix} \mathcal{U}_1(a) & \mathcal{U}_2(a) \\ \mathcal{U}_1(b) & \mathcal{U}_2(b) \end{vmatrix} \neq 0$$

No guarantee that this will be satisfied but will generally find a solution.

From our previous solutions, $\mathcal{U}_+(x)$, $\mathcal{U}_-(x)$ where $\mathcal{U}_+(b) = 0$, $\mathcal{U}_-(a) = 0$

$$y = y_p(x) + y_a \frac{\mathcal{U}_+(x)}{\mathcal{U}_+(a)} + y_b \frac{\mathcal{U}_-(x)}{\mathcal{U}_-(b)}$$

Boundary conditions for boundary value problems

For a general second order equation have several possible boundary conditions

① Dirichlet

The value of the function is specified at the boundary
$\Rightarrow$ we just worked this case out

For a second order eqn have two degrees of freedom associated with the two homogeneous solutions so usually have a solution.

② Neumann

Specify the derivative of the function at each boundary. Can generalize the Green's function results to solve this problem. Again, generally have two degrees of freedom so can find solution

③ Cauchy

Specify both function and derivative on boundaries. Not enough degrees of freedom to satisfy.